1. A certain $2 \times 2$ matrix $A$ has eigenvalues 1 and 2. The eigenspaces are shown in the picture below.
   a) Draw $Av$, $A^2v$, and $Aw$.
   b) Compute the limit of $A^n v/\|A^n v\|$ as $n \to \infty$.

   Solution.

   As $n \to \infty$, the vector $A^n v/\|A^n v\|$ tends to $\frac{1}{\sqrt{2}}(-1,-1)$.

2. A certain diagonalizable $2 \times 2$ matrix $A$ is equal to $CDC^{-1}$, where $C$ has columns $v_1, v_2$ pictured below, and $D = \begin{pmatrix} 1/3 & 0 \\ 0 & 1/2 \end{pmatrix}$.
   a) Draw $C^{-1} v$ on the left.
   b) Draw $DC^{-1} v$ on the left.
c) Draw $A\nu = CDC^{-1}\nu$ on the right.

d) What happens to $A^n\nu$ as $n \to \infty$?

\[ C^{-1}\nu \]
\[ DC^{-1}\nu \]

As $n \to \infty$, the vector $A^n\nu$ approaches 0.

3. Compute the following complex numbers.

\[ \begin{align*}
\text{a)} \ (1 + i) + (2 - i) & \quad \text{b)} \ (1 + i)(2 - i) & \quad \text{c)} \ 2 - i & \quad \text{d)} \ \frac{1 + i}{2 - i} \\
\text{e)} \ |1 + i| & \quad \text{f)} \ 2e^{2\pi i/3} & \quad \text{g)} \ 5e^{3\pi i}
\end{align*} \]

Solution.
4. Express each complex number in polar coordinates \(r e^{i\theta}\).

\[
\begin{align*}
a) & \quad 1 + i \\
b) & \quad -\frac{1 + i \sqrt{3}}{2} \\
c) & \quad -\sqrt{3} - 3i \\
d) & \quad \frac{1}{1 + i} \\
e) & \quad (1 - i \sqrt{3})^n
\end{align*}
\]

**Solution.**

\[
\begin{align*}
a) & \quad \sqrt{2} e^{\pi i/4} \\
b) & \quad e^{2\pi i/3} \\
c) & \quad 2\sqrt{3} e^{-2\pi i/3} \\
d) & \quad \frac{1}{\sqrt{2}} e^{-\pi i/4} \\
e) & \quad 2^n e^{-n\pi i/3}
\end{align*}
\]

5. For which numbers \(\theta\) is \(e^{i\theta} = 1\)? What about \(-1\)?

**Solution.**

We have \(e^{i\theta} = 1\) if and only if \(\theta\) is an integer multiple of \(2\pi\). We have \(e^{i\theta} = -1\) if and only if \(\theta\) is \(\pi\) plus an integer multiple of \(2\pi\).

6. For each matrix \(A\) and each vector \(x\), decide if \(x\) is an eigenvector of \(A\), and if so, find the eigenvalue \(\lambda\).

\[
\begin{align*}
a) & \quad \begin{pmatrix} 2 & 3 \\ -3 & 2 \end{pmatrix}, \quad \begin{pmatrix} i \\ 1 \end{pmatrix} \\
b) & \quad \begin{pmatrix} -4 & 13 & 13 \\ 2 & -2 & -4 \\ -4 & 8 & 10 \end{pmatrix}, \quad \begin{pmatrix} 1 + 5i \\ -2i \\ 4i \end{pmatrix} \\
c) & \quad \begin{pmatrix} 1 & 1 & 1 \\ -1 & -3 & -3 \\ -2 & 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 2 + i \\ 1 \\ -i \end{pmatrix}
\end{align*}
\]

Careful! It is difficult to recognize by inspection if two complex vectors are (complex) scalar multiples of each other.

**Solution.**

a) yes, \(\lambda = 2 - 3i\)  
   b) yes, \(\lambda = 1 + i\)  
   c) not an eigenvector

7. For each \(2 \times 2\) matrix \(A\), \(i)\) compute the characteristic polynomial, \(ii)\) find all (real and complex) eigenvalues, and \(iii)\) find a basis for each eigenspace, using Problem 3 on Homework 10 when applicable. \(iv)\) Is the matrix diagonalizable (over the complex numbers)? If so, find an invertible matrix \(C\) and a diagonal matrix \(D\) such that \(A = CDC^{-1}\).

\[
\begin{align*}
a) & \quad \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \\
b) & \quad \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \\
c) & \quad \begin{pmatrix} -3 & 5 \\ -10 & 7 \end{pmatrix}
\end{align*}
\]

**Solution.**
a) $p(\lambda) = \lambda^2 - 2\lambda + 2$. The eigenvalues are $\lambda = 1 + i$ and $\bar{\lambda} = 1 - i$, with eigenspaces spanned by $v = \begin{pmatrix} i \\ 1 \end{pmatrix}$ and $\bar{v} = \begin{pmatrix} -i \\ 1 \end{pmatrix}$, respectively. The matrix is equal to $CDC^{-1}$ for
\[ C = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}, \quad D = \begin{pmatrix} 1 + i & 0 \\ 0 & 1 - i \end{pmatrix}. \]

b) $p(\lambda) = \lambda^2 - 2\lambda + 5$. The eigenvalues are $\lambda = 1 + 2i$ and $\bar{\lambda} = 1 - 2i$, with eigenspaces spanned by $v = \begin{pmatrix} i \\ 1 \end{pmatrix}$ and $\bar{v} = \begin{pmatrix} -i \\ 1 \end{pmatrix}$, respectively. The matrix is equal to $CDC^{-1}$ for
\[ C = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}, \quad D = \begin{pmatrix} 1 + 2i & 0 \\ 0 & 1 - 2i \end{pmatrix}. \]

c) $p(\lambda) = \lambda^2 - 4\lambda + 29$. The eigenvalues are $\lambda = 2 + 5i$ and $\bar{\lambda} = 2 - 5i$, with eigenspaces spanned by $v = \begin{pmatrix} 1 \\ 1+i \end{pmatrix}$ and $\bar{v} = \begin{pmatrix} 1 \\ 1-i \end{pmatrix}$, respectively. The matrix is equal to $CDC^{-1}$ for
\[ C = \begin{pmatrix} 1 & 1 \\ 1+i & 1-i \end{pmatrix}, \quad D = \begin{pmatrix} 2 + 5i & 0 \\ 0 & 2 - 5i \end{pmatrix}. \]

8. Diagonalize the following matrix over the complex numbers:
\[ A = \begin{pmatrix} 1 & 4 & -6 \\ -6 & 7 & -22 \\ -2 & 1 & -5 \end{pmatrix}. \]

One eigenvalue is $\lambda = -1$.

**Solution.**

We compute the characteristic polynomial to be
\[ p(\lambda) = -(\lambda + 1)(\lambda^2 - 4\lambda + 5). \]

The real eigenvalue is $\lambda_1 = -1$, and the complex eigenvalues are $\lambda_2 = 2 + i$ and $\bar{\lambda}_2 = 2 - i$. An eigenvector with eigenvalue $\lambda_1 = -1$ is $w_1 = (-1, 2, 1)$. Computing an eigenvector with eigenvalue $\lambda_2$ is tedious, as it involves Gaussian elimination with complex numbers. There’s no way around it: the result is $w_2 = (-3 - i, 1 - i, 1)$. The last eigenvector $\bar{w}_2 = (-3 + i, 1 + i, 1)$ comes for free. It follows that $A = CDC^{-1}$ for
\[ C = \begin{pmatrix} -1 & -3 - i & -3 + i \\ 2 & 1 - i & 1 + i \\ 1 & 1 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 + i & 0 \\ 0 & 0 & 2 - i \end{pmatrix}. \]

9. A certain forest contains a population of rabbits and a population of foxes. If there are $r_n$ rabbits and $f_n$ foxes in year $n$, then
\[ r_{n+1} = 3r_n - f_n, \]
\[ f_{n+1} = r_n + 2f_n. \]
in other words, each rabbit produces three baby rabbits on average, but there is some loss due to predation by foxes; each fox produces two babies on average, but this is increased with ample prey.

a) Let \( v_n = \begin{pmatrix} r_n \\ f_n \end{pmatrix} \). Find a matrix \( A \) such that \( v_{n+1} = Av_n \).

b) Find an eigenbasis of \( A \). (The eigenvectors and eigenvalues will be complex.)

[Hint: Part d) will be easier if you choose the eigenvectors with first coordinate equal to 1.]

c) Suppose that \( r_0 = 2 \) and \( f_0 = 1 \). Find closed formulas for \( r_n \) and \( f_n \). Find a formula for \( r_n \) involving only real numbers. (This latter formula can involve an arctan.)

d) In this model, the populations do not stabilize. How many years will it take for the foxes to eat all of the rabbits?

In general, any \( 2 \times 2 \) difference equation with a complex eigenvalue will exhibit oscillation centered at zero. This phenomenon can be described explicitly, but is beyond the scope of this course.

Solution.

a) \( A = \begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix} \)

b) The eigenvalues of \( A \) are \( \lambda = \frac{1}{2}(5 + i \sqrt{3}) \) and \( \bar{\lambda} = \frac{1}{2}(5 - i \sqrt{3}) \), with eigenspaces spanned by \( v = \left( 1, \frac{1}{2}(1 - i \sqrt{3}) \right) \) and \( \bar{v} = \left( 1, \frac{1}{2}(1 + i \sqrt{3}) \right) \), respectively.

c) We have \( v_0 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} = v + \bar{v} \), so

\[
\begin{pmatrix} r_n \\ f_n \end{pmatrix} = v_n = A^n v_0 = \lambda^n v + \bar{\lambda}^n \bar{v}
\]

\[
= \left( \frac{5 + i \sqrt{3}}{2} \right)^n \left( \frac{1}{2}(1 - i \sqrt{3}) \right) + \left( \frac{5 - i \sqrt{3}}{2} \right)^n \left( \frac{1}{2}(1 + i \sqrt{3}) \right).
\]

To find a formula for \( r_n \) involving only real numbers, we rewrite

\[
r_n = \frac{1}{2^n} \left( (5 + i \sqrt{3})^n + (5 - i \sqrt{3})^n \right) = \frac{1}{2^n} 2 \text{Re}((5 + i \sqrt{3})^n).
\]

To compute this quantity, we put \( 5 + i \sqrt{3} \) in polar form:

\[
5 + i \sqrt{3} = 2 \sqrt{7} e^{i \theta} \quad \theta = \arctan \left( \frac{\sqrt{3}}{5} \right) \approx 0.3335.
\]

Therefore,

\[
r_n = \frac{1}{2^n} 2 \text{Re}((2 \sqrt{7})^n e^{i n \theta}) = 2(\sqrt{7})^n \cos(n \theta).
\]

d) We have \( r_n < 0 \) as soon as \( \cos(n \theta) < 0 \), which happens when \( n = 5 \).
b) Find a non-diagonal $2 \times 2$ matrix such that 1 is an eigenvalue with algebraic multiplicity 2.

Solution.

a) To say that $\lambda$ is an eigenvalue of $A$ with geometric multiplicity $n$ means that $\text{Nul}(A - \lambda I_n) = \mathbb{R}^n$, so $A - \lambda I_n = 0$, i.e., $A = \lambda I_n$.

b) For instance, \( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \).

11. Find examples of real $2 \times 2$ matrices $A$ with the following properties.

a) $A$ is invertible and diagonalizable over the real numbers.

b) $A$ is invertible but not diagonalizable over the complex numbers.

c) $A$ is diagonalizable over the real numbers but not invertible.

d) $A$ is neither invertible nor diagonalizable over the complex numbers.

This shows that invertibility and diagonalizability have nothing to do with each other.

Solution.

There are many examples; these are the canonical ones.

\[
\begin{array}{cccc}
\text{a)} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{b)} & \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} & \text{c)} & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \text{b)} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}
\end{array}
\]

12. Let $A$ be an $n \times n$ matrix.

a) Show that the product of the (real and complex) eigenvalues, counted with algebraic multiplicity, is equal to $\det(A)$.

b) [Optional] Show that the sum of the (real and complex) eigenvalues, counted with algebraic multiplicity, is equal to $\text{Tr}(A)$.

(Both of these are identities involving the characteristic polynomial of $A$.)

Solution.

Let $p(\lambda)$ be the characteristic polynomial of $A$. Over the complex numbers we can factor

\[ p(\lambda) = (-1)^n(\lambda - \lambda_1) \cdots (\lambda - \lambda_n), \]

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues, counted with algebraic multiplicity (that is, an eigenvalue with algebraic multiplicity 2 is counted twice).

a) Substituting 0 for $\lambda$ yields

\[ \det(A) = p(0) = (-1)^n(-\lambda_1) \cdots (-\lambda_n) = (-1)^{2n}\lambda_1 \cdots \lambda_n = \lambda_1 \cdots \lambda_n. \]

b) The $\lambda^{n-1}$-coefficient of $p(\lambda)$ is $-(-1)^n(\lambda_1 + \cdots + \lambda_n) = (-1)^{n-1}(\lambda_1 + \cdots + \lambda_n)$. But we know that $p(\lambda) = (-1)^n\lambda^n + (-1)^{n-1}\text{Tr}(A)\lambda^{n-1} + \cdots + \det(A)$. 

13. For each matrix in Problem 5(a)–(c) on Homework 10, compute the algebraic and geometric multiplicity of each eigenvalue. What does your answer say about diagonalizability? **Optional:** do (d)–(g) as well.

**Solution.**

a) All three eigenvalues have \(AM=GM=1\), which means it’s diagonalizable.

b) The eigenvalue 1 has \(AM=GM=1\) and the eigenvalue \(-1\) has \(AM=GM=2\), which means it’s diagonalizable.

c) The eigenvalue 1 has \(AM=3\) and \(GM=1\), so it’s not diagonalizable.

d) All three eigenvalues have \(AM=GM=1\), which means it’s diagonalizable.

e) The eigenvalue 2 has \(AM=GM=3\).

f) The eigenvalue 1 has \(AM=3\) and \(GM=1\), so it’s not diagonalizable.

g) All four eigenvalues have \(AM=GM=1\), which means it’s diagonalizable.

14. Give an example of each of the following, or explain why no such example exists. All matrices should have real entries.

a) A \(3 \times 3\) matrix with eigenvalues 0, 1, 2, and corresponding eigenvectors

\[
\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.
\]

b) A \(4 \times 4\) matrix having eigenvalue 2 with algebraic multiplicity 2 and geometric multiplicity 3.

c) A \(3 \times 3\) matrix with one complex (non-real) eigenvalue and two real eigenvalues.

d) A \(2 \times 2\) matrix \(A\) such that \(A^2\) is diagonalizable over the real numbers but \(A\) is not diagonalizable, even over the complex numbers.

[**Hint:** try a nonzero matrix \(A\) such that \(A^2 = 0\).]

**Solution.**

a) Does not exist: these vectors are linearly dependent.

b) Does not exist: \(AM \geq GM\).

c) Does not exist: a real \(3 \times 3\) matrix with a complex (non-real) eigenvalue has two complex eigenvalues and one real eigenvalue.

d) One such example is \(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\).

15. Decide if each statement is true or false, and explain why.

a) If \(A\) and \(B\) are diagonalizable \(n \times n\) matrices, then so is \(AB\).
b) An $n \times n$ matrix with $n$ (different) eigenvalues is diagonalizable.

c) An $n \times n$ matrix is diagonalizable if it has $n$ eigenvalues, counted with algebraic multiplicity.

d) Any $2 \times 2$ real matrix with a complex (non-real) eigenvalue is diagonalizable over the complex numbers.

e) Any $3 \times 3$ real matrix with a complex (non-real) eigenvalue is diagonalizable over the complex numbers.

f) Any $4 \times 4$ real matrix with a complex (non-real) eigenvalue is diagonalizable over the complex numbers.

g) Any $2 \times 2$ real matrix has a real eigenvalue.

h) Any $3 \times 3$ real matrix has a real eigenvalue.

i) Any $n \times n$ matrix has a (real or complex) eigenvalue.

j) If the characteristic polynomial of $A$ is $-(\lambda^3 - 1) = -(\lambda^2 + \lambda + 1)(\lambda - 1)$, then the 1-eigenspace of $A$ is a line.

Solution.

a) False: take $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

b) True.

c) False: any matrix has $n$ eigenvalues counted with algebraic multiplicity.

d) True: it has distinct eigenvalues $\lambda$ and $\bar{\lambda}$.

e) True: it has distinct eigenvalues $\lambda_1$, $\lambda_2$, and $\bar{\lambda}_2$.

f) False: for instance, 

\[
\begin{pmatrix}
1 & 1 & 1 & 1 \\
-1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 \\
0 & 0 & -1 & 1 \\
\end{pmatrix}
\]

is not diagonalizable.

g) False.

h) True: any odd-degree polynomial has a real root.

i) True: any non-constant polynomial has a complex root.

j) True: the algebraic multiplicity is 1, so the geometric multiplicity is also 1.