Homework #10

due Wednesday, November 3, at 11:59pm

1. For each matrix A and each vector v, decide if v is an eigenvector of A, and if so, find the eigenvalue λ.

$$\textbf{a)}\ \begin{pmatrix} -20 & 42 & 58 \\ 1 & -1 & -3 \\ -1 & 18 & 26 \end{pmatrix}, \begin{pmatrix} 1 \\ 5 \\ -2 \end{pmatrix} \qquad \textbf{b)}\ \begin{pmatrix} 2 & 3 & 0 \\ -5 & 4 & 2 \\ 3 & 3 & 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$$

$$\textbf{c)}\ \begin{pmatrix} -7 & 32 & -76 \\ 7 & -22 & 59 \\ 3 & -11 & 28 \end{pmatrix}, \begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix} \qquad \textbf{d)}\ \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

$$\textbf{e)}\ \begin{pmatrix} -3 & 2 & -3 \\ 3 & -3 & -2 \\ -4 & 2 & -3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

2. Suppose that A is an $n \times n$ matrix such that $Av = 2v$ for some $v \neq 0$. Let C be any invertible matrix. Consider the matrices

$$\textbf{a)}\ A^{-1} \qquad \textbf{b)}\ A + 2I_n \qquad \textbf{c)}\ A^3 \qquad \textbf{d)}\ CAC^{-1}.$$

Show that v is an eigenvector of **a)–c)** and that Cv is as eigenvector of **d)**, and find the eigenvalues.

3. Here is a handy trick for computing eigenvectors of a 2×2 matrix.

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a 2×2 matrix with eigenvalue λ.

a) Explain why $\begin{pmatrix} -b \\ a-\lambda \end{pmatrix}$ and $\begin{pmatrix} d-\lambda \\ -c \end{pmatrix}$ are λ-eigenvectors of A if they are nonzero.

b) Suppose that A has another eigenvalue $\lambda' \neq \lambda$. Explain why $\begin{pmatrix} a-\lambda \\ c \end{pmatrix}$ and $\begin{pmatrix} b \\ d-\lambda \end{pmatrix}$ (the columns of $A - \lambda I_2$) are λ'-eigenvectors of A if they are nonzero. (No, this is not a typo.)

[**Hint:** Show $A(A - \lambda I_2) = \lambda'(A - \lambda I_2)$ by showing that $(A - \lambda' I_2)(A - \lambda I_2) = 0$: multiply by a vector expanded in an eigenbasis.]

Hence you can usually compute all eigenvectors of a 2×2 matrix very quickly.

4. For each 2×2 matrix A, **i)** compute the characteristic polynomial using the formula $p(\lambda) = \lambda^2 - \mathrm{Tr}(A)\lambda + \det(A)$. Use this to **ii)** find all real eigenvalues, and **iii)** find a basis for each eigenspace, using Problem 3 when applicable. **iv)** Draw and label each eigenspace. **v)** Is the matrix diagonalizable (over the real numbers)? If so, find an invertible matrix C and a diagonal matrix D such that $A = CDC^{-1}$.

$$\textbf{a)}\ \begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix} \qquad \textbf{b)}\ \begin{pmatrix} -1 & 1 \\ -9 & 5 \end{pmatrix} \qquad \textbf{c)}\ \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \qquad \textbf{d)}\ \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \qquad \textbf{e)}\ \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

5. For each matrix A, **i)** find all real eigenvalues of A, and **ii)** find a basis for each eigenspace. **iii)** Is the matrix diagonalizable (over the real numbers)? If so, find an invertible matrix C and a diagonal matrix D such that $A = CDC^{-1}$.

At least one eigenvalue is provided; use that and synthetic division (or a computer algebra system) to find the others.

a) $\begin{pmatrix} -1 & 7 & 5 \\ 0 & 1 & -2 \\ 0 & 1 & 4 \end{pmatrix}$, $\lambda = -1$ **b)** $\begin{pmatrix} 7 & 12 & 12 \\ -8 & -13 & -12 \\ 4 & 6 & 5 \end{pmatrix}$, $\lambda = 1$

c) $\begin{pmatrix} 6 & 2 & 3 \\ -14 & -7 & -12 \\ 1 & 2 & 4 \end{pmatrix}$, $\lambda = 1$

Optional (if you want more practice):

d) $\begin{pmatrix} -11 & -54 & 10 \\ -2 & -7 & 2 \\ -21 & -90 & 20 \end{pmatrix}$, $\lambda = 1$ **e)** $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$, $\lambda = 2$

f) $\begin{pmatrix} 13 & 18 & -18 \\ -12 & -17 & 18 \\ -4 & -6 & 7 \end{pmatrix}$, $\lambda = 1$ **g)** $\begin{pmatrix} -10 & 28 & -18 & -76 \\ -1 & 9 & -6 & -2 \\ 4 & -8 & 7 & 26 \\ 0 & 2 & -2 & 4 \end{pmatrix}$, $\begin{matrix} \lambda_1 = 1 \\ \lambda_2 = 2 \end{matrix}$

6. Consider the matrix

$$A = \begin{pmatrix} 4 & -3 & 0 \\ 2 & -1 & 0 \\ 1 & -1 & 1 \end{pmatrix}.$$

a) Find a diagonal matrix D and an invertible matrix C such that $A = CDC^{-1}$.

b) Find a *different* diagonal matrix D' and a *different* invertible matrix C' such that $A = C'D'C'^{-1}$.

[**Hint:** Try re-ordering the eigenvalues.]

7. Compute the matrix with eigenvalues $0, 1, 2$ and corresponding eigenvectors

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}.$$

(There is only one such matrix.)

8. **a)** Show that A and A^T have the same eigenvalues.

b) Give an example of a 2×2 matrix A such that A and A^T do not share any eigenvectors.

c) A *stochastic matrix* is a matrix with nonnegative entries whose columns sum to 1. Explain why 1 is an eigenvalue of a stochastic matrix.

[**Hint:** show that $(1, 1, \ldots, 1)$ is an eigenvector of A^T.]

9. **a)** Find all eigenvalues of the matrix

$$\begin{pmatrix} 1 & -1 & 2 & 3 & 4 \\ 0 & 3 & -1 & -2 & -5 \\ 0 & 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

b) Explain how to find the eigenvalues of any triangular matrix.

10. Let A and B be $n \times n$ matrices, and let $v_1, \ldots, v_n$ be a basis of $\mathbf{R}^n$.

a) Suppose that each v_i is an eigenvector of both A and B. Show that $AB = BA$.

b) Suppose that each v_i is an eigenvector of both A and B with the same eigenvalue. Show that $A = B$.

[**Hint:** To show two matrices are equal, try multiplying them by any vector, expanded in your eigenbasis. Alternatively, use the matrix form of diagonalization.]

11. Let A be an $n \times n$ matrix, and let C be an invertible $n \times n$ matrix. Prove that the characteristic polynomial of CAC^{-1} equals the characteristic polynomial of A.

In particular, A and CAC^{-1} have the same eigenvalues, the same determinant, and the same trace. They are called *similar* matrices.

12. Recall that an *orthogonal matrix* is a square matrix with orthonormal columns.

a) Prove that any real eigenvalue of an orthogonal matrix Q is ± 1.

b) Let L be the line through $(1,1,1)$, and let $R_L = I_3 - 2P_L$ be the *reflection* over the plane $x + y + z = 0$ (the orthogonal complement of Span$\{(1,1,1)\}$). Compute R_L, and diagonalize R_L without doing any work.

13. The *Fibonacci numbers* are defined recursively as follows:

$$F_0 = 0, \qquad F_1 = 1, \qquad F_{n+2} = F_{n+1} + F_n \ \ (n \geq 0).$$

The first few Fibonacci numbers are $0, 1, 1, 2, 3, 5, 8, 13, \ldots$ In this problem, you will find a closed formula (as opposed to a recursive formula) for the nth Fibonacci number using diagonalization.

a) Let $v_n = \binom{F_{n+1}}{F_n}$, so $v_0 = \binom{1}{0}$, $v_1 = \binom{1}{1}$, etc. Find a transition matrix A such that $v_{n+1} = Av_n$ for all $n \geq 0$.

b) Show that the eigenvalues of A are $\lambda_1 = \frac{1}{2}(1+\sqrt{5})$ and $\lambda_2 = \frac{1}{2}(1-\sqrt{5})$, with corresponding eigenvectors $w_1 = \binom{-1}{\lambda_2}$ and $w_2 = \binom{-1}{\lambda_1}$.

[**Hint:** Just show $Aw_i = \lambda_i w_i$ using the relations $\lambda_1\lambda_2 = -1$ and $\lambda_1 + \lambda_2 = 1$.]

c) Find x_1, x_2 such that $v_0 = x_1 w_1 + x_2 w_2$. (It helps to write x_1, x_2 in terms of λ_1, λ_2.)

d) Multiply $v_0 = x_1 w_1 + x_2 w_2$ by A^n to show that
$$F_n = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2}.$$

e) Use this formula to explain why F_{n+1}/F_n approaches the golden ratio when n is large.

14. Pretend that there are three Red Box kiosks in Durham. Let x_t, y_t, z_t be the number of copies of Prognosis Negative at each of the three kiosks, respectively, on day t. Suppose in addition that a customer renting a movie from kiosk i will return the movie the next day to kiosk j, with the following probabilities:

Returning to kiosk	Renting from kiosk 1	Renting from kiosk 2	Renting from kiosk 3
1	30%	40%	50%
2	30%	40%	30%
3	40%	20%	20%

For instance, a customer renting from kiosk 3 has a 50% probability of returning it to kiosk 1.

a) Let $v_t = (x_t, y_t, z_t)$. Find the state change matrix A such that $v_{t+1} = Av_t$.

b) Find a basis of $\mathbf{R}^3$ consisting of eigenvectors of A. What are the eigenvalues? [**Hint:** A is a stochastic matrix, so you know one eigenvalue by Problem 8(c).]

c) If you start with a total of 1 000 copies of Prognosis Negative, how many of them will eventually end up at each kiosk?

This is an example of a stochastic process, and is an important application of eigenvalues and eigenvectors.

15. Let $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$. Find a closed formula for A^n: that is, an expression of the form
$$A^n = \begin{pmatrix} a_{11}(n) & a_{12}(n) \\ a_{21}(n) & a_{22}(n) \end{pmatrix},$$
where $a_{ij}(n)$ is a function of n.

16. Give an example of each of the following, or explain why no such example exists.

a) An invertible matrix with characteristic polynomial $p(\lambda) = -\lambda^3 + 2\lambda^2 + 3\lambda$.

b) A 2×2 orthogonal matrix with no real eigenvalues.

17. Suppose that A is a square matrix such that A^k is the zero matrix for some $k > 0$. Show that 0 is the only eigenvalue of A.

18. Decide if each statement is true or false, and explain why.

a) If v, w are eigenvectors of a matrix A, then so is $v + w$.

b) An eigenvalue of $A + B$ is the sum of an eigenvalue of A and an eigenvalue of B.

c) An eigenvalue of AB is the product of an eigenvalue of A and an eigenvalue of B.

d) If $Ax = \lambda x$ for some vector x, then λ is an eigenvalue of A.

e) A matrix with eigenvalue 0 is not invertible.

f) The eigenvalues of A are equal to the eigenvalues of a row echelon form of A.

g) If v, w are eigenvectors of A with different eigenvalues, then $\{v, w\}$ is linearly independent.