Homework #1
due Monday, August 30, at 11:59pm

1. Consider the vectors
   \[ v = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad w = \begin{pmatrix} -1 \\ 1 \end{pmatrix}. \]
   Draw the 16 linear combinations \( cv + dw \) (\( c, d = -1, 0, 1, 2 \)) in the \( xy \)-plane.

2. Certain vectors \( v, w \) in \( \mathbb{R}^2 \) are drawn below. Express each of \( b_1, b_2, b_3, b_4, b_5 \) as a linear combination of \( v, w \).

![Diagram](image)

3. If
   \[ v + w = \begin{pmatrix} -4 \\ 1 \end{pmatrix} \quad \text{and} \quad v - w = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \]
   compute and draw the vectors \( v \) and \( w \).

4. Consider the vectors
   \[ u = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}, \quad v = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}, \quad w = \begin{pmatrix} 8 \\ -6 \\ -2 \end{pmatrix}. \]
   a) Compute \( u + v + w \) and \( u + 2v - w \).
   b) Find numbers \( x \) and \( y \) such that \( w = xu + yv \).
   c) Explain why every linear combination of \( u, v, w \) is also a linear combination of \( u \) and \( v \) only.
   d) The sum of the coordinates of any linear combination of \( u, v, w \) is equal to _____?
   e) Find a vector in \( \mathbb{R}^3 \) that is not a linear combination of \( u, v, w \).
5. Consider the vectors

\[ u = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \]

Draw a picture of all of the linear combinations \( au + bv \) for real numbers \( a, b \) satisfying \( 0 \leq a \leq 1 \) and \( 0 \leq b \leq 1 \).

6. Consider the vectors pointing towards the numbers on a clock:

\[ \text{2:00} \]

a) What is the sum of all twelve of these vectors?

b) If the 2:00 vector is removed, why do the remaining vectors add to 8:00?

7. Find two different triples \((x, y, z)\) such that

\[ x \begin{pmatrix} 1 \\ 2 \end{pmatrix} + y \begin{pmatrix} 1 \\ -2 \end{pmatrix} + z \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}. \]

How many such triples are there?

8. Decide if each statement is true or false, and explain why.

a) The vector \( \frac{1}{2} v \) is a linear combination of \( v \) and \( w \).

b) \( \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \).

c) If \( v, w \) are two vectors in \( \mathbb{R}^2 \), then any other vector \( b \) in \( \mathbb{R}^2 \) is a linear combination of \( v \) and \( w \).

9. Consider the following vectors:

\[ u = \begin{pmatrix} -0.6 \\ .8 \end{pmatrix}, \quad v = \begin{pmatrix} 4 \\ 3 \end{pmatrix}, \quad w = \begin{pmatrix} 1 \\ 2 \end{pmatrix}. \]

a) Compute the lengths \( \|u\|, \|v\|, \) and \( \|w\| \).

b) Compute the lengths \( \|2u\|, \| -v\|, \) and \( \|3w\| \).

c) Find the unit vectors in the directions of \( u, v, \) and \( w \).
d) Compute the dot products \( u \cdot v, u \cdot w, \) and \( v \cdot w \). Verify that they are the same as \( v \cdot u, w \cdot u, \) and \( w \cdot v \), respectively.

e) Check the Schwartz inequalities \(|u \cdot v| \leq \|u\| \|v\|\) and \(|v \cdot w| \leq \|v\| \|w\|\).

f) Find the angles between \( u \) and \( v \) and between \( v \) and \( w \).

g) Find the distance from \( v \) to \( w \).

h) Find unit vectors \( u', v', w' \) orthogonal to \( u, v, w \), respectively.

10. Suppose that \( v \) and \( w \) are unit vectors. Compute the following dot products (your answers will be actual numbers):

\[
\begin{align*}
\text{a) } v \cdot (-v) & \quad \text{b) } (v + w) \cdot (v - w) & \quad \text{c) } (v + 2w) \cdot (v - 2w).
\end{align*}
\]

11. Decide if each statement is true or false, and explain why.

\[
\begin{align*}
a) \text{ If } u = (1, 1, 1) \text{ is orthogonal to } v \text{ and to } w, \text{ then } v \text{ is parallel to } w. \\
b) \text{ If } u \text{ is orthogonal to } v + w \text{ and to } v - w, \text{ then } u \text{ is orthogonal to } v \text{ and } w. \\
c) \text{ If } u \text{ and } v \text{ are orthogonal unit vectors then } \|u - v\| = \sqrt{2}. \\
d) \text{ If } \|u\|^2 + \|v\|^2 = \|u + v\|^2, \text{ then } u \text{ and } v \text{ are orthogonal.}
\end{align*}
\]

12. Find nonzero vectors \( v \) and \( w \) that are orthogonal to \((1, 1, 1)\) and to each other.

13. What is the length of the vector \( v = (1, 1, \ldots, 1) \) in \( n \) dimensions?

14. If \( \|v\| = 5 \) and \( \|w\| = 3 \), what are the smallest and largest possible values of \( \|v - w\| \)? What are the smallest and largest possible values of \( v \cdot w \)? Justify your answer using the algebra of dot products.

15. \[ \begin{align*}
a) \text{ If } v \cdot w < 0, \text{ what does that say about the angle between } v \text{ and } w? \\
b) \text{ Find three vectors } u, v, w \text{ in the } xy\text{-plane such that } u \cdot v < 0, u \cdot w < 0, \text{ and } v \cdot w < 0.
\end{align*} \]

16. Compute the following matrix-vector products using both the row-first and column-first methods. If the product is not defined, explain why.

\[
\begin{align*}
\begin{pmatrix} 2 \\ 5 \\
-3 \\ -1 \end{pmatrix} \begin{pmatrix} 1 \\ -3 \\ 3 \\ 2 \end{pmatrix} & \quad \begin{pmatrix} 1 \\ 0 \\ 3 \\ 2 \end{pmatrix} \begin{pmatrix} -2 \\ 0 \\ -1 \\ -2 \end{pmatrix} & \quad \begin{pmatrix} 7 \\ 3 \\ 3 \\ 1 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \\ -3 \\ 1 \end{pmatrix} \\
\begin{pmatrix} -2 \\ 2 \\ 4 \\ 1 \end{pmatrix} \begin{pmatrix} -1 \\ -2 \end{pmatrix} & \quad \begin{pmatrix} 2 \\ 6 \\ -1 \end{pmatrix} \begin{pmatrix} 5 \\ -1 \\ 0 \end{pmatrix} & \quad \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 2 \\ 6 \\ -1 \end{pmatrix}
\end{align*}
\]
17. Suppose that \( u = (x, y, z) \) and \( v = (a, b, c) \) are vectors satisfying \( 2u + 3v = 0 \). Find a nonzero vector \( w \) in \( \mathbb{R}^2 \) such that
\[
\begin{pmatrix} x & a \\ y & b \\ z & c \end{pmatrix} w = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.
\]

18. Consider the matrices
\[
A = \begin{pmatrix} 2 & 1 & -1 \\ 4 & -4 & 2 \end{pmatrix} \quad B = \begin{pmatrix} 5 & 3 & 2 \\ 1 & -1 & 2 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}
\]
\[
D = \begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix} \quad E = \begin{pmatrix} -3 & 5 \end{pmatrix}.
\]
Compute the following expressions. If the result is not defined, explain why.
- a) \(-3A\)
- b) \(B - 3A\)
- c) \(AC\)
- d) \(B^2\)
- e) \(A + 2B\)
- f) \(C - E\)
- g) \(EB\)
- h) \(D^2\)

19. Compute the product
\[
\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 2 & 1 & -1 \\ 4 & -1 & 2 \end{pmatrix}
\]
in three ways:
- a) Using the “column first” method.
- b) Using the “rows first” method.
- c) Using the outer product form.

20. Consider the matrices
\[
A = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 1 \\ -1 & h \end{pmatrix}.
\]
What value(s) of \( h \), if any, will make \( AB = BA \)?

21. Consider the matrices
\[
A = \begin{pmatrix} 1 & -3 \\ 2 & 5 \end{pmatrix} \quad B = \begin{pmatrix} -4 & -8 \\ 5 & 8 \end{pmatrix} \quad C = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}.
\]
Verify that \( AC = BC \) and yet \( A \neq B \).

22. For the following matrices \( A \) and \( B \), compute \( AB, A^T, B^T, B^T A^T \), and \((AB)^T\). Which of these matrices are equal and why? Why can't you compute \( A^T B^T \)?
\[
A = \begin{pmatrix} 1 & 2 \\ -2 & -1 \end{pmatrix} \quad B = \begin{pmatrix} 2 & 1 & -1 \\ 4 & -1 & 2 \end{pmatrix}.
\]

23. In the table below, a linear system is expressed as a system of equations, as a matrix equation, or as an augmented matrix. Fill in the blank entries.
Consider the following system of equations:

\[ \begin{align*}
3x_1 + 2x_2 + 4x_3 &= 9 \\
-x_1 + 4x_3 &= 2
\end{align*} \]

\[
\begin{pmatrix}
3 & -5 \\
2 & 4 \\
-1 & 1
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix} =
\begin{pmatrix}
1 \\
1 \\
2
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & 0 & 1 & 1 & | & 2 \\
0 & 3 & -1 & -2 & | & 4 \\
1 & -3 & -4 & -3 & | & 2 \\
6 & 5 & -1 & -8 & | & 1
\end{pmatrix}
\]

**24.** Consider the following system of equations:

\[ \begin{align*}
x_1 - 2x_2 + x_3 &= 1 \\
-2x_1 + 5x_2 + 5x_3 &= 2 \\
3x_1 - 7x_2 - 7x_3 &= 2
\end{align*} \]

a) Use row operations to eliminate \(x_1\) from all but the first equation.

b) Use row operations to modify the system so that \(x_2\) only appears in the first and second equations (and \(x_1\) still only appears in the first).

c) Solve for \(x_3\), then for \(x_2\), then for \(x_1\). What is the solution?

**25.** The matrix below can be transformed into row echelon form using exactly two row operations. What are they?

\[
\begin{pmatrix}
2 & 4 & -2 & 4 \\
-1 & -2 & 1 & -2 \\
0 & 2 & 0 & 3
\end{pmatrix}
\]