MATH 218D-1
MIDTERM EXAMINATION 3

Please **read all instructions** carefully before beginning.

- Do not open this test booklet until you are directed to do so.
- You have 75 minutes to complete this exam.
- If you finish early, go back and check your work.
- The graders will only see the work on the **printed pages**. You may use other scratch paper, but the graders will not see anything written there.
- You may use a **calculator** for doing arithmetic, but you should not need one. All other materials and aids are strictly prohibited.
- For full credit you must **show your work** so that your reasoning is clear, unless otherwise indicated.
- Do not spend too much time on any one problem. Read them all through first and attack them in an order that allows you to make the most progress.
- Good luck!
Problem 1. [20 points]

Compute the determinants of the following matrices.

a) \( \det \begin{pmatrix}
1 & 2 & 3 & 4 & 5 \\
1 & 2 & 3 & 4 & 5 \\
1 & 2 & 3 & 4 & 5 \\
1 & 2 & 3 & 4 & 5 \\
1 & 2 & 3 & 4 & 5 \\
\end{pmatrix} \) = 

b) \( \det \begin{pmatrix}
5 & 0 & 0 \\
-3 & 0 & 0 \\
8 & 5 & -1 \\
\end{pmatrix} \) = 

c) \( \det \begin{pmatrix}
2 & 1 & 1 & 0 \\
0 & 2 & 1 & 1 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 7 \\
0 & 0 & 0 & 1 \\
\end{pmatrix} \) = 

d) \( \det \begin{pmatrix}
1 & -2 & 3 \\
2 & 0 & -6 \\
1 & 0 & -3 \\
\end{pmatrix} \) = 

Problem 2. [20 points]

Consider the matrix
\[ A = \frac{1}{10} \begin{pmatrix} 11 & -3 \\ -3 & 19 \end{pmatrix}. \]

a) Find the eigenvalues of \( A \) and orthonormal eigenvectors.

\[ \lambda_1 = \quad w_1 = \begin{pmatrix} \quad \\ \end{pmatrix} \]
\[ \lambda_2 = \quad w_2 = \begin{pmatrix} \quad \\ \end{pmatrix} \]

b) Draw the eigenspaces of \( A \) in the grid below, and label them with the corresponding eigenvalues. Be precise!

c) Vectors \( v \) and \( w \) are shown in the picture. Draw and label the vectors
\[ v_\infty = \lim_{k \to \infty} \frac{A^k v}{\|A^k v\|} \quad \text{and} \quad w_\infty = \lim_{k \to \infty} \frac{A^k w}{\|A^k w\|}. \]
Problem 3. [20 points]

In this problem, you need not explain your answers; just write them in the spaces provided.

Let $A$ be an $n \times n$ matrix.

a) Which one of the following statements is correct?

1. An eigenvector of $A$ is a vector $v$ such that $Av = \lambda v$ for a nonzero scalar $\lambda$.
2. An eigenvector of $A$ is a nonzero vector $v$ such that $Av = \lambda v$ for a scalar $\lambda$.
3. An eigenvector of $A$ is a nonzero scalar $\lambda$ such that $Av = \lambda v$ for some vector $v$.
4. An eigenvector of $A$ is a nonzero vector $v$ such that $Av = \lambda v$ for a nonzero scalar $\lambda$.

b) Which one of the following statements is not correct?

1. An eigenvalue of $A$ is a scalar $\lambda$ such that $A - \lambda I$ is not invertible.
2. An eigenvalue of $A$ is a scalar $\lambda$ such that $(A - \lambda I)v = 0$ has a solution.
3. An eigenvalue of $A$ is a scalar $\lambda$ such that $Av = \lambda v$ for a nonzero vector $v$.
4. An eigenvalue of $A$ is a scalar $\lambda$ such that $\det(A - \lambda I) = 0$.

c) Which of the following $3 \times 3$ matrices are necessarily diagonalizable over the real numbers? (List all that apply.)

1. A matrix with three distinct real eigenvalues.
2. A symmetric matrix with two real eigenvalues.
3. A matrix with a real eigenvalue $\lambda$ of algebraic multiplicity 2, such that the $\lambda$-eigenspace has dimension 2.
4. A matrix with a real eigenvalue $\lambda$ such that the $\lambda$-eigenspace has dimension 2.

d) Give an example of a $2 \times 2$ matrix that is neither invertible nor diagonalizable.

\[
\begin{pmatrix}
& \\
& \\
& \\
\end{pmatrix}
\]
Problem 4.  

20 points

a) Compute the characteristic polynomial of the following matrix. Do not factor it!

\[
\begin{pmatrix}
3 & 0 & 1 \\
-1 & 2 & 0 \\
2 & 2 & 4
\end{pmatrix}
\implies p(\lambda) = \text{[Enter polynomial here]}
\]
b) Consider the matrix

\[
A = \begin{pmatrix}
3 & 7 & -11 \\
6 & 22 & -33 \\
4 & 14 & -21
\end{pmatrix}.
\]

The eigenvalues of \( A \) are \( \lambda_1 = 1 \) and \( \lambda_2 = 2 \). Find an invertible matrix \( C \) and a diagonal matrix \( D \) such that \( A = C D C^{-1} \).

\[
C = \begin{pmatrix}
\end{pmatrix}, \\
D = \begin{pmatrix}
\end{pmatrix}
\]

c) In part b), is it possible to find an orthogonal matrix \( Q \) and a diagonal matrix \( D \) such that \( A = QDQ^T \)? Why or why not?
Problem 5. [20 points]

Consider the following initial value problem:
\[ u'_1 = 2u_1 - u_2 \quad u_1(0) = 2 \]
\[ u'_2 = 3u_1 - u_2 \quad u_2(0) = 3. \]

a) Let \( u(t) = (u_1(t), u_2(t)) \). Find the matrix \( A \) such that \( u' = Au \).

\[
A = \begin{pmatrix}
\end{pmatrix}
\]

b) Find the eigenvalues of \( A \).

\[
\lambda = \quad \bar{\lambda} = \quad \lambda = \quad \bar{\lambda} =
\]

c) For each eigenvalue \( \lambda_i \), find the corresponding eigenvector \( w_i \) whose first coordinate is 1.

\[
w = \begin{pmatrix}
1
\end{pmatrix}
\quad \bar{w} = \begin{pmatrix}
1
\end{pmatrix}
\]
d) Express $u(0) = (2, 3)$ as a linear combination of the eigenvectors you found in c).

\[
\begin{pmatrix} 2 \\ 3 \end{pmatrix} = w + \bar{w}.
\]

e) Solve the initial value problem $u' = Au$, $u(0) = (2, 3)$. Your answer should involve only real numbers.

\[
\begin{align*}
u_1 &= \\
u_2 &= 
\end{align*}
\]