

**MATH 218D-1**  
**PRACTICE MIDTERM EXAMINATION 3**

<b>Name</b>		<b>Duke Email</b>	
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Please **read all instructions** carefully before beginning.

- Do not open this test booklet until you are directed to do so.
- You have 75 minutes to complete this exam.
- If you finish early, go back and check your work.
- The graders will only see the work on the **printed pages**. You may use other scratch paper, but the graders will not see anything written there.
- You may use a **calculator** for doing arithmetic, but you should not need one. All other materials and aids are strictly prohibited.
- For full credit you must **show your work** so that your reasoning is clear, unless otherwise indicated.
- Do not spend too much time on any one problem. Read them all through first and attack them in an order that allows you to make the most progress.
- Good luck!

This is a practice exam. It is meant to be similar in format, length, and difficulty to the real exam. It is **not** meant as a comprehensive list of study problems. I recommend completing the practice exam in 75 minutes, without notes or distractions.

## Problem 1.

[20 points]

a) Verify that the symmetric matrix

$$S = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 6 & -2 \\ 0 & -2 & 5 \end{pmatrix}.$$

is positive-definite without finding its eigenvalues.

b) Compute the characteristic polynomial of the matrix in part a). (Do not factor it.)

c) Consider the symmetric matrix

$$S = \begin{pmatrix} 2 & 1 & -4 \\ 1 & 2 & 4 \\ -4 & 4 & 5 \end{pmatrix}.$$

Find an orthogonal matrix  $Q$  and a diagonal matrix  $D$  such that  $S = QDQ^T$ . The eigenvalues of  $S$  are 9, 3, and  $-3$ .

## Problem 2.

[20 points]

Consider the difference equation

$$\begin{aligned}x_{n+1} &= 2x_n - y_n & x_0 &= 1 \\y_{n+1} &= \frac{3}{2}x_n - \frac{1}{2}y_n & y_0 &= 2.\end{aligned}$$

a) Find a matrix  $A$  such that

$$A \begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix}.$$

b) Find the eigenvalues of  $A$ , and find corresponding eigenvectors.

c) Find a formula for  $\begin{pmatrix} x_n \\ y_n \end{pmatrix}$  in terms of  $n$ .

d) What is  $\lim_{n \rightarrow \infty} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$ ?

### Problem 3.

[10 points]

Solve the following initial value problem:

$$\begin{aligned}u_1' &= 2u_1 - u_2 & u_1(0) &= 1 \\u_2' &= \frac{3}{2}u_1 - \frac{1}{2}u_2 & u_2(0) &= 2.\end{aligned}$$

## Problem 4.

[20 points]

Give examples of matrices with each of the following properties. If no such matrix exists, explain why. *All matrices in this problem have real entries.*

a) A symmetric matrix satisfying

$$S \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} \quad \text{and} \quad S \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix}.$$

b) A  $2 \times 2$  matrix whose 1-eigenspace is the line  $x + 2y = 0$  and whose 2-eigenspace is the line  $x + 3y = 0$ .

c) A  $2 \times 2$  matrix that is neither invertible nor diagonalizable.

d) A  $2 \times 2$  non-invertible matrix with eigenvalue  $2 + 3i$ .

e) A  $2 \times 2$  matrix  $A$  that is diagonalizable over  $\mathbf{R}$ , such that  $A^2$  is not diagonalizable.

## Problem 5.

[5 points]

Let  $A$  be an  $n \times n$  matrix with characteristic polynomial

$$p(\lambda) = \lambda(\lambda - 2)(\lambda - 3)^2.$$

Which of the following can you determine from this information? (Select all that apply.)

- |                              |                                    |
|------------------------------|------------------------------------|
| (1) The number $n$ .         | (5) Whether $A$ is symmetric.      |
| (2) The trace of $A$ .       | (6) Whether $A$ is diagonalizable. |
| (3) The determinant of $A$ . | (7) The eigenvalues of $A$ .       |
| (4) The rank of $A$ .        |                                    |

## Problem 6.

[10 points]

A certain diagonalizable  $2 \times 2$  matrix  $A$  has eigenvalues 1 and 2, with eigenspaces drawn below.

- Draw  $Ax$  and  $Ay$  on the diagram.
- Draw the vector  $w = \lim_{n \rightarrow \infty} A^n x / \|A^n x\|$ : that is, eventually  $A^n x$  points in the direction of the unit vector  $w$ . (Let's say that 1cm on your paper is one unit.)

