Please read all instructions carefully before beginning.

- Do not open this test booklet until you are directed to do so.
- You have 75 minutes to complete this exam.
- If you finish early, go back and check your work.
- The graders will only see the work on the printed pages. You may use other scratch paper, but the graders will not see anything written there.
- You may use a calculator for doing arithmetic, but you should not need one. All other materials and aids are strictly prohibited.
- For full credit you must show your work so that your reasoning is clear, unless otherwise indicated.
- Do not spend too much time on any one problem. Read them all through first and attack them in an order that allows you to make the most progress.
- Good luck!

This is a practice exam. It is meant to be similar in format, length, and difficulty to the real exam. It is not meant as a comprehensive list of study problems. I recommend completing the practice exam in 75 minutes, without notes or distractions.
Problem 1. [20 points]

a) Verify that the symmetric matrix

\[ S = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 6 & -2 \\ 0 & -2 & 5 \end{pmatrix}. \]

is positive-definite without finding its eigenvalues.

b) Compute the characteristic polynomial of the matrix in part a). (Do not factor it.)

c) Consider the symmetric matrix

\[ S = \begin{pmatrix} 2 & 1 & -4 \\ 1 & 2 & 4 \\ -4 & 4 & 5 \end{pmatrix}. \]

Find an orthogonal matrix \( Q \) and a diagonal matrix \( D \) such that \( S = QDQ^T \). The eigenvalues of \( S \) are 9, 3, and \(-3\).

Solution.

a) This can be accomplished by finding the \( LU \) decomposition:

\[ S = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & -2 \\ 0 & 0 & 3 \end{pmatrix}. \]

b) \[ p(\lambda) = -\lambda^3 + 12^2 - 33\lambda + 6 \]

c) \[ Q = \begin{pmatrix} -1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ 2/\sqrt{6} & 0 & 1/\sqrt{3} \end{pmatrix} \quad D = \begin{pmatrix} 9 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -3 \end{pmatrix} \]
Problem 2. [20 points]

Consider the difference equation
\[
\begin{align*}
x_{n+1} &= 2x_n - y_n \quad x_0 = 1 \\
y_{n+1} &= \frac{3}{2}x_n - \frac{1}{2}y_n \quad y_0 = 2.
\end{align*}
\]

a) Find a matrix $A$ such that
\[A \begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix}.
\]

b) Find the eigenvalues of $A$, and find corresponding eigenvectors.

c) Find a formula for $\begin{pmatrix} x_n \\ y_n \end{pmatrix}$ in terms of $n$.

d) What is $\lim_{n \to \infty} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$?

Solution.

a) The matrix is $A = \begin{pmatrix} 2 & -1 \\ 3/2 & -1/2 \end{pmatrix}$.

b) The eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 1/2$, and corresponding eigenvectors are $w_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $w_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

c) We have $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = -w_1 + w_2$, so
\[
\begin{pmatrix} x_n \\ y_n \end{pmatrix} = -A^n w_1 + A^n w_2 = -w_1 + \frac{1}{2^n} w_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{2^n} \begin{pmatrix} 2 \\ 3 \end{pmatrix}.
\]

d) The limit is $-w_1 = -\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. 

Problem 3. [10 points]

Solve the following initial value problem:

\[ u'_1 = 2u_1 - u_2 \quad u_1(0) = 1 \]
\[ u'_2 = \frac{3}{2}u_1 - \frac{1}{2}u_2 \quad u_2(0) = 2. \]

Solution.

\[ u_1(t) = -e^t + 2e^{t/2} \]
\[ u_2(t) = -e^t + 3e^{t/2}. \]
Problem 4. [20 points]

Give examples of matrices with each of the following properties. If no such matrix exists, explain why. All matrices in this problem have real entries.

a) A symmetric matrix satisfying

\[\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix}.\]

b) A 2 × 2 matrix whose 1-eigenspace is the line \(x + 2y = 0\) and whose 2-eigenspace is the line \(x + 3y = 0\).

c) A 2 × 2 matrix that is neither invertible nor diagonalizable.

d) A 2 × 2 non-invertible matrix with eigenvalue \(2 + 3i\).

e) A 2 × 2 matrix \(A\) that is diagonalizable over \(\mathbb{R}\), such that \(A^2\) is not diagonalizable.

Solution.

a) Does not exist: eigenvectors with different eigenvalues would have to be orthogonal.

b) This matrix satisfies

\[A = \begin{pmatrix} -2 & -3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} -2 & -3 \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 4 & 6 \\ -1 & -1 \end{pmatrix}.\]

c) One example is \(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\).

d) Does not exist: the other eigenvalue would be \(2 - 3i\), so 0 is not an eigenvalue.

e) Does not exist: if \(A = CDC^{-1}\) then \(A^2 = CD^2C^{-1}\).
Problem 5. [5 points]

Let $A$ be an $n \times n$ matrix with characteristic polynomial

$$p(\lambda) = \lambda(\lambda - 2)(\lambda - 3)^2.$$ 

Which of the following can you determine from this information? (Select all that apply.)

1. The number $n$.
2. The trace of $A$.
3. The determinant of $A$.
4. The rank of $A$.
5. Whether $A$ is symmetric.
6. Whether $A$ is diagonalizable.
7. The eigenvalues of $A$.

Solution.

You can determine (1), (2), (3), (4), and (7).

1. $n = \deg(p) = 4$.
2. $\text{Tr}(A)$ is the sum of the eigenvalues (with multiplicity), which is $2 + 3 + 3 = 8$.
3. $\det(A)$ is the product of the eigenvalues (with multiplicity), which is 0.
4. The null space is the 0-eigenspace, which has algebraic multiplicity 1, hence also geometric multiplicity 1. Therefore $\dim \text{Nul}(A) = 1$, so $\text{rank}(A) = 4 - 1 = 3$.
5. You can’t tell. Both of these matrices have characteristic polynomial $p(\lambda)$:

$$
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 \\
0 & 0 & 3 & 0 \\
0 & 0 & 0 & 3
\end{pmatrix}
\quad
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 \\
0 & 0 & 3 & 1 \\
0 & 0 & 0 & 3
\end{pmatrix},
$$

6. You can’t tell. The first matrix above is diagonal, and the second is not diagonalizable.
7. The eigenvalues are 0, 2, and 3.
Problem 6. [10 points]

A certain diagonalizable $2 \times 2$ matrix $A$ has eigenvalues 1 and 2, with eigenspaces drawn below.

a) Draw $Ax$ and $Ay$ on the diagram.

b) Draw the vector $w = \lim_{n \to \infty} A^n x / \|A^n x\|$: that is, eventually $A^n x$ points in the direction of the unit vector $w$. (Let’s say that 1cm on your paper is one unit.)