Please read all instructions carefully before beginning.

- Do not open this test booklet until you are directed to do so.
- You have 75 minutes to complete this exam.
- If you finish early, go back and check your work.
- The graders will only see the work on the printed pages. You may use other scratch paper, but the graders will not see anything written there.
- You may use a calculator for doing arithmetic, but you should not need one. All other materials and aids are strictly prohibited.
- For full credit you must show your work so that your reasoning is clear, unless otherwise indicated.
- Do not spend too much time on any one problem. Read them all through first and attack them in an order that allows you to make the most progress.
- Good luck!

\[
\begin{bmatrix}
\cos 90^\circ & \sin 90^\circ \\
-sin 90^\circ & \cos 90^\circ
\end{bmatrix}
\begin{bmatrix}
a_1 \\
a_2
\end{bmatrix}
= \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

[Hint: this is a joke.]
Problem 1. [20 points]

Consider the plane 

\[ V = \{(x, y, z) : x - y + 2z = 0\}. \]

a) Find a basis for \( V \).

b) Find an orthogonal basis for \( V \).

c) Use the projection formula and your answer to part b) to compute the orthogonal projection \( b_V \) of the vector \( b = (1, 1, -3) \) onto \( V \).

\[ b_V = \begin{pmatrix} \end{pmatrix} \]

d) Find a basis for \( V^\perp \).

e) Find an orthogonal basis of \( \mathbb{R}^3 \) containing the basis vectors you found in b).
Problem 2. [20 points]

Consider the matrix

\[
A = \begin{pmatrix}
1 & 2 & 5 \\
-1 & 1 & -4 \\
-1 & 4 & -3 \\
1 & -4 & 7 \\
1 & 2 & 1
\end{pmatrix}
\]

a) Find the QR decomposition of \( A \). You should get

\[
R = \begin{pmatrix}
p \sqrt{5} & -p & 4p \\
0 & 6 & -2 \\
0 & 0 & 4
\end{pmatrix}
\]

\[
Q = \begin{pmatrix}
\end{pmatrix}
\]
b) Solve \( R \hat{x} = Q^T \begin{pmatrix} 2 \\ -2 \\ 4 \\ -3 \\ 3 \end{pmatrix} \) to find the least-squares solution of \( Ax = \begin{pmatrix} 2 \\ -2 \\ 4 \\ -3 \\ 3 \end{pmatrix} \).

\[
\hat{x} = \begin{pmatrix} \\
\end{pmatrix}
\]

c) Compute the matrix \( P_V \) for projection onto \( V = \text{Col}(A) \).

\[
P_V = \begin{pmatrix} \\
\end{pmatrix}
\]
Problem 3. [15 points]

Consider the data points

\[b_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad b_1 = \begin{pmatrix} 1 \\ 8 \end{pmatrix}, \quad b_2 = \begin{pmatrix} 3 \\ 8 \end{pmatrix}, \quad b_3 = \begin{pmatrix} 4 \\ 20 \end{pmatrix}\]

drawn below.

a) Find the matrix \( A \) such that the least-squares solution \( \hat{x} = (C, D) \) of

\[
A \begin{pmatrix} C \\ D \end{pmatrix} = b = \begin{pmatrix} 0 \\ 8 \\ 8 \\ 20 \end{pmatrix}
\]
gives the coefficients of the best-fit line \( y = Cx + D \).

b) Find the equation of the best-fit line by computing the least-squares solution of the above equation. Graph this line in the above grid.

\[y = \square x + \square\]
c) Compute the minimized vector $b_{V\perp}$. What does $b_{V\perp}$ represent in the original best-fit problem? (Here $V = \text{Col}(A)$.)

\[
b_{V\perp} = \begin{pmatrix} \ \\
\end{pmatrix}
\]

d) What is the best-fit line among all lines passing through the origin?

\[y = \square x\]
Problem 4. [12 points]

A line $V$ and a vector $b$ are drawn below. Draw and label:

a) The orthogonal projection $b_V$.

b) The projection onto the orthogonal complement $b_{V\perp}$.

c) The vector $b - 2b_{V\perp}$. 
Problem 5.  

Consider the vectors

\[
\begin{align*}
\mathbf{v}_1 &= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\
\mathbf{v}_2 &= \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \\
\mathbf{v}_3 &= \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix} \\
\mathbf{v}_4 &= \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}
\end{align*}
\]

and the subspace \( W = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\} \).

[Hint: in this problem it is helpful, but not necessary, to use the fact that \( \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \) is orthogonal.]

a) Find a linear relation among \( \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4 \).

b) What is the dimension of \( W \)?  
\[ \dim(W) = \square \]

c) List all subsets of \( \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\} \) that form a basis for \( W \).
Problem 6. [16 points]

All of the following statements are false. Find a counterexample.

a) If $Q$ has orthonormal columns, then $QQ^T$ is the identity matrix.

b) If $V$ and $W$ are subspaces of $\mathbb{R}^n$ and every vector in $V$ is orthogonal to every vector in $W$, then $V = W^\perp$.

c) A matrix with orthogonal columns has full row rank.

d) If $A$ is an $m \times n$ matrix and $A^T A$ is invertible, then $A$ has rank $m$.

e) If $Q$ has orthogonal columns, then $\|Qx\| = \|x\|$ for any vector $x$. 