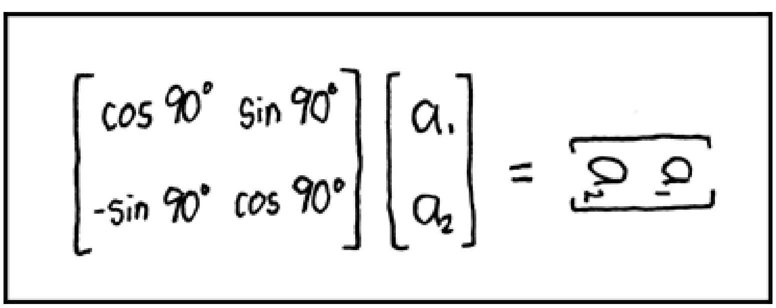
MATH 218D-1
MIDTERM EXAMINATION 2

Name		**Duke Email**	@duke.edu

Please **read all instructions** carefully before beginning.

- Do not open this test booklet until you are directed to do so.
- You have 75 minutes to complete this exam.
- If you finish early, go back and check your work.
- The graders will only see the work on the **printed pages**. You may use other scratch paper, but the graders will not see anything written there.
- You may use a **calculator** for doing arithmetic, but you should not need one. All other materials and aids are strictly prohibited.
- For full credit you must **show your work** so that your reasoning is clear, unless otherwise indicated.
- Do not spend too much time on any one problem. Read them all through first and attack them in an order that allows you to make the most progress.
- Good luck!



[Hint: this is a joke.]

Problem 1. [20 points]

Consider the plane

$$V = \{(x, y, z) \ : \ x - y + 2z = 0\}.$$

a) Find a basis for V.

$$\left\{ \qquad \right\}$$

b) Find an orthogonal basis for V.

$$\left\{ \qquad \right\}$$

c) Use the projection formula and your answer to part **b)** to compute the orthogonal projection b_V of the vector $b = (1, 1, -3)$ onto V.

$$b_V = \left(\qquad \right)$$

d) Find a basis for $V^\perp$.

$$\left\{ \qquad \right\}$$

e) Find an orthogonal basis of $\mathbf{R}^3$ containing the basis vectors you found in **b)**.

$$\left\{ \qquad \right\}$$

Problem 2.

[20 points]

Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 5 \\ -1 & 1 & -4 \\ -1 & 4 & -3 \\ 1 & -4 & 7 \\ 1 & 2 & 1 \end{pmatrix}.$$

a) Find the QR decomposition of A. You should get $R = \begin{pmatrix} \sqrt{5} & -\sqrt{5} & 4\sqrt{5} \\ 0 & 6 & -2 \\ 0 & 0 & 4 \end{pmatrix}$.

$$Q = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$$

b) Solve $R\widehat{x} = Q^T \begin{pmatrix} 2 \\ -2 \\ 4 \\ -3 \\ 3 \end{pmatrix}$ to find the least-squares solution of $Ax = \begin{pmatrix} 2 \\ -2 \\ 4 \\ -3 \\ 3 \end{pmatrix}$.

$$\widehat{x} = \begin{pmatrix} \\ \\ \\ \end{pmatrix}$$

c) Compute the matrix P_V for projection onto $V = \text{Col}(A)$.

$$P_V = \begin{pmatrix} \\ \\ \\ \end{pmatrix}$$

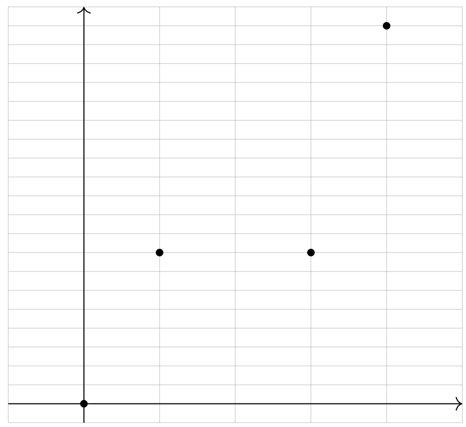
Problem 3.

[15 points]

Consider the data points

$$b_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad b_1 = \begin{pmatrix} 1 \\ 8 \end{pmatrix} \quad b_2 = \begin{pmatrix} 3 \\ 8 \end{pmatrix} \quad b_3 = \begin{pmatrix} 4 \\ 20 \end{pmatrix}$$

drawn below.



a) Find the matrix A such that the least-squares solution $\widehat{x} = (C, D)$ of

$$A\begin{pmatrix} C \\ D \end{pmatrix} = b = \begin{pmatrix} 0 \\ 8 \\ 8 \\ 20 \end{pmatrix}$$

gives the coefficients of the best-fit line $y = Cx + D$.

$$A = \begin{pmatrix} \quad & \quad \\ & \\ & \end{pmatrix}$$

b) Find the equation of the best-fit line by computing the least-squares solution of the above equation. Graph this line in the above grid.

$y = \square\, x + \square$

c) Compute the minimized vector $b_{V^\perp}$. What does $b_{V^\perp}$ represent in the original best-fit problem? (Here $V = \mathrm{Col}(A)$.)

$$b_{V^\perp} = \begin{pmatrix} \\ \\ \\ \end{pmatrix}$$

d) What is the best-fit line among all lines *passing through the origin*?

$$y = \boxed{}\,x$$

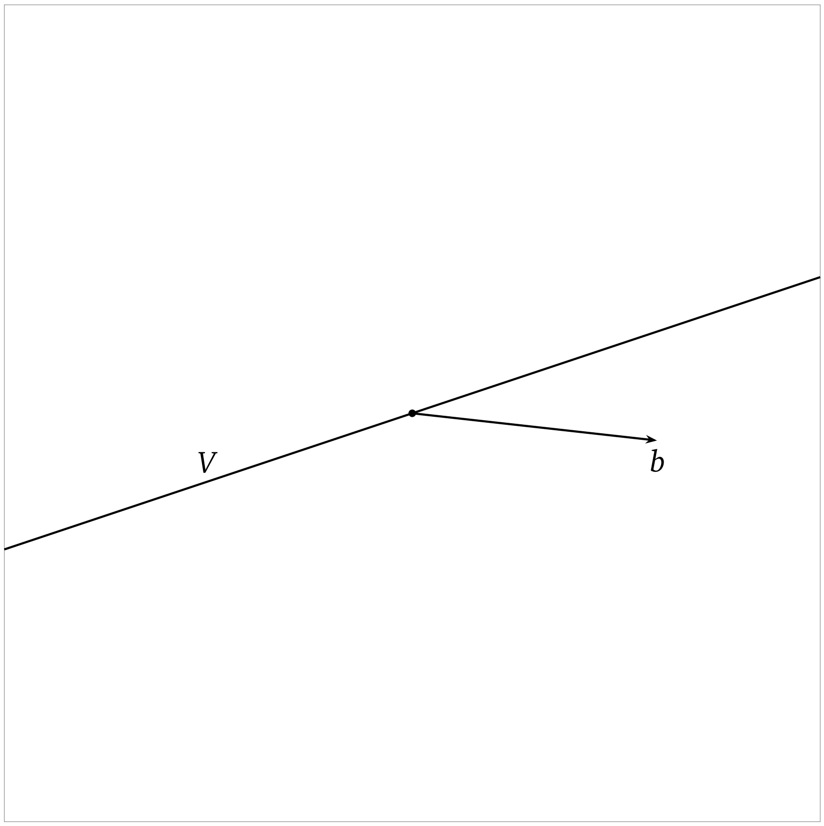
Problem 4. [12 points]

A line V and a vector b are drawn below. Draw *and label*:

a) The orthogonal projection b_V.

b) The projection onto the orthogonal complement $b_{V^\perp}$.

c) The vector $b - 2b_{V^\perp}$.



Problem 5. [15 points]

Consider the vectors

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \qquad v_3 = \begin{pmatrix} 2 \\ 0 \\ -2 \\ 0 \end{pmatrix} \qquad v_4 = \begin{pmatrix} 4 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

and the subspace $W = \text{Span}\{v_1, v_2, v_3, v_4\}$.

[**Hint:** in this problem it is helpful, but not necessary, to use the fact that $\{v_1, v_2, v_3\}$ is orthogonal.]

a) Find a linear relation among v_1, v_2, v_3, v_4.

b) What is the dimension of W?

$\dim(W) = \square$

c) List all subsets of $\{v_1, v_2, v_3, v_4\}$ that form a basis for W.

Problem 6. [16 points]

All of the following statements are false. Find a counterexample.

a) If Q has orthonormal columns, then QQ^T is the identity matrix.

b) If V and W are subspaces of $\mathbf{R}^n$ and every vector in V is orthogonal to every vector in W, then $V = W^{\perp}$.

c) A matrix with orthogonal columns has full row rank.

d) If A is an $m \times n$ matrix and A^TA is invertible, then A has rank m.

e) If Q has orthogonal columns, then $\|Qx\| = \|x\|$ for any vector x.