Please read all instructions carefully before beginning.

- Do not open this test booklet until you are directed to do so.
- You have 75 minutes to complete this exam.
- If you finish early, go back and check your work.
- The graders will only see the work on the printed pages. You may use other scratch paper, but the graders will not see anything written there.
- You may use a calculator for doing arithmetic, but you should not need one. All other materials and aids are strictly prohibited.
- For full credit you must show your work so that your reasoning is clear, unless otherwise indicated.
- Do not spend too much time on any one problem. Read them all through first and attack them in an order that allows you to make the most progress.
- Good luck!

\[
\begin{bmatrix}
\cos 90^\circ & \sin 90^\circ \\
-\sin 90^\circ & \cos 90^\circ 
\end{bmatrix}
\begin{bmatrix}
\alpha_1 \\
\alpha_2 
\end{bmatrix} = \frac{\varphi}{\varphi}
\]

[Hint: this is a joke.]
Problem 1.  

Consider the plane 

\[ V = \{ (x, y, z) : x - y + 2z = 0 \} \].

a) Find a basis for \( V \).

b) Find an orthogonal basis for \( V \).

c) Use the projection formula and your answer to part b) to compute the orthogonal projection \( b_V \) of the vector \( b = (1, 1, -3) \) onto \( V \).

d) Find a basis for \( V^\perp \).

e) Find an orthogonal basis of \( \mathbb{R}^3 \) containing the basis vectors you found in b).

Solution.

a) There are many answers. If you find the solutions of \( x - y + 2z = 0 \) in parametric vector form, you get 

\[ \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \right\} \].

b) Running Gram–Schmidt on the above vectors gives 

\[ \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\} \].

c) \( b_V = \begin{pmatrix} 1 \\ -3 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -3 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} \).

d) Since \( V = \text{Nul} \begin{pmatrix} 1 & -1 & 2 \end{pmatrix} \), the orthogonal complement \( V^\perp \) is the row space of \( \begin{pmatrix} 1 & -1 & 2 \end{pmatrix} \):

\[ \left\{ \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \right\} \].

e) We just add the vectors in b) and d):

\[ \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \\ 2 \end{pmatrix} \right\} \].

(You could also notice that \( b - b_V = (-1, 1, -2) \) spans \( V^\perp \).)
Problem 2.  

Consider the matrix 

\[ A = \begin{pmatrix}
1 & 2 & 5 \\
-1 & 1 & -4 \\
-1 & 4 & -3 \\
1 & -4 & 7 \\
1 & 2 & 1
\end{pmatrix}. \]

a) Find the QR decomposition of \( A \). You should get \( R = \begin{pmatrix}
\sqrt{5} & -\sqrt{5} & 4\sqrt{5} \\
0 & 6 & -2 \\
0 & 0 & 4
\end{pmatrix}. \)

b) Solve \( R\hat{x} = Q^T \begin{pmatrix} 2 \\ -2 \\ 4 \\ -3 \\ 3 \end{pmatrix} \) to find the least-squares solution of \( Ax = \begin{pmatrix} 2 \\ -2 \\ 4 \\ -3 \\ 3 \end{pmatrix}. \)

c) Compute the matrix \( P_v \) for projection onto \( V = \text{Col}(A) \).

Solution.

a) \[ Q = \begin{pmatrix}
1/\sqrt{5} & 1/2 & 1/2 \\
-1/\sqrt{5} & 0 & 0 \\
-1/\sqrt{5} & 1/2 & 1/2 \\
1/\sqrt{5} & -1/2 & 1/2 \\
1/\sqrt{5} & 1/2 & -1/2
\end{pmatrix} \]

b) \[ \hat{x} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \]

c) \[ P_v = QQ^T = \frac{1}{10} \begin{pmatrix}
7 & -2 & 3 & 2 & 2 \\
-2 & 2 & 2 & -2 & -2 \\
3 & 2 & 7 & -2 & -2 \\
2 & -2 & -2 & 7 & -3 \\
2 & -2 & -2 & -3 & 7
\end{pmatrix} \]
Problem 3.  

Consider the data points
\[ b_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad b_1 = \begin{pmatrix} 1 \\ 8 \end{pmatrix}, \quad b_2 = \begin{pmatrix} 3 \\ 8 \end{pmatrix}, \quad b_3 = \begin{pmatrix} 4 \\ 20 \end{pmatrix} \]
drawn below.

a) Find the matrix \( A \) such that the least-squares solution \( Ax = (C, D) \) of
\[
A \begin{pmatrix} C \\ D \end{pmatrix} = b = \begin{pmatrix} 0 \\ 8 \\ 8 \\ 20 \end{pmatrix}
\]
gives the coefficients of the best-fit line \( y = Cx + D \).

b) Find the equation of the best-fit line by computing the least-squares solution of the above equation. Graph this line in the above grid.

c) Compute the minimized vector \( b_{V^\perp} \). What does \( b_{V^\perp} \) represent in the original best-fit problem? (Here \( V = \text{Col}(A) \).)

d) What is the best-fit line among all lines passing through the origin?
Solution.

a) \[ A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 3 & 1 \\ 4 & 1 \end{pmatrix} \]

b) \[ \hat{x} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} \quad \text{\(\Rightarrow\)} \quad y = 4x + 1 \]

c) The minimized vector is

\[ b_{\perp} = b - A\hat{x} = \begin{pmatrix} 0 \\ 8 \\ 8 \\ 20 \end{pmatrix} - \begin{pmatrix} 1 \\ 5 \\ 13 \\ 17 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ -5 \\ 3 \end{pmatrix}. \]

This is the vector of vertical distances from the data points to the graph of the best-fit line, drawn in red in the picture.

d) Using \( y = Cx \) means solving the least-squares problem

\[ \begin{pmatrix} 0 \\ 1 \\ 3 \\ 4 \end{pmatrix} C = \begin{pmatrix} 0 \\ 8 \\ 8 \\ 20 \end{pmatrix} \quad \text{\(\Rightarrow\)} \quad C = \frac{56}{13}. \]

The best-fit line is \( y = \frac{56}{13}x \).
Problem 4. [12 points]

A line $V$ and a vector $b$ are drawn below. Draw and label:

a) The orthogonal projection $b_V$.

b) The projection onto the orthogonal complement $b_{V\perp}$.

c) The vector $b - 2b_{V\perp}$. 
Problem 5. [15 points]

Consider the vectors

\[ v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 2 \\ 0 \\ -2 \\ 0 \end{pmatrix}, \quad v_4 = \begin{pmatrix} 4 \\ 0 \\ 0 \\ 0 \end{pmatrix} \]

and the subspace \( W = \text{Span}\{v_1, v_2, v_3, v_4\} \).

[Hint: in this problem it is helpful, but not necessary, to use the fact that \( \{v_1, v_2, v_3\} \) is orthogonal.]

a) Find a linear relation among \( v_1, v_2, v_3, v_4 \).

b) What is the dimension of \( W \)?

c) List all subsets of \( \{v_1, v_2, v_3, v_4\} \) that form a basis for \( W \).

Solution.

a) \( v_1 + v_2 + v_3 - v_4 = 0 \)

b) \( \text{dim}(W) = 3 \)

c) \( \{v_1, v_2, v_3\}, \{v_1, v_2, v_4\}, \{v_1, v_3, v_4\}, \{v_2, v_3, v_4\} \)
Problem 6.  

[16 points] 

All of the following statements are false. Find a counterexample.  

a) If $Q$ has orthonormal columns, then $QQ^T$ is the identity matrix.  

b) If $V$ and $W$ are subspaces of $\mathbb{R}^n$ and every vector in $V$ is orthogonal to every vector in $W$, then $V = W^\perp$.  

c) A matrix with orthogonal columns has full row rank.  

d) If $A$ is an $m \times n$ matrix and $A^TA$ is invertible, then $A$ has rank $m$.  

e) If $Q$ has orthogonal columns, then $\|Qx\| = \|x\|$ for any vector $x$.  

Solution.

a) Any non-square matrix with orthonormal columns is a counterexample.

b) For example, $V = \text{Span}\{(1, 0, 0)\}$ and $W = \text{Span}\{(0, 1, 0)\}$.

c) Any non-square matrix with orthogonal columns is a counterexample.

d) Any non-square matrix with full column rank is a counterexample.

e) Choose a matrix $Q$ with orthogonal columns whose first column does not have length 1; then $\|Qe_1\| \neq 1 = \|e_1\|$.