Please read all instructions carefully before beginning.

• Do not open this test booklet until you are directed to do so.
• You have 75 minutes to complete this exam.
• If you finish early, go back and check your work.
• You may use a calculator for doing arithmetic, but you should not need one. All other materials and aids are strictly prohibited.
• For full credit you must show your work so that your reasoning is clear, unless otherwise indicated.
• Do not spend too much time on any one problem. Read them all through first and attack them in an order that allows you to make the most progress.
• Good luck!

This is a practice exam. It is meant to be similar in format, length, and difficulty to the real exam. It is not meant as a comprehensive list of study problems. I recommend completing the practice exam in 75 minutes, without notes or distractions.
Problem 1. [15 points]

Consider the vectors

\[
v_1 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \quad v_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \quad v_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.
\]

a) Is \( \{v_1, v_2, v_3\} \) linearly independent? If not, find a linear relation.

b) Compute the dimension of \( \text{Span}\{v_1, v_2, v_3\} \).

c) Write \((2, 6, -2)\) as a linear combination of \(v_1, v_2, v_3\).
Problem 2. [20 points]

Find a basis of the orthogonal complement of each of the following subspaces.

a) Nul \begin{pmatrix} 1 & 0 & 2 & 3 \\ 2 & 1 & 4 & 1 \end{pmatrix}

b) Col \begin{pmatrix} 1 & 2 & -4 \\ 0 & -1 & 3 \\ 3 & 0 & 6 \\ 4 & -1 & 11 \end{pmatrix}

c) The subspace of all vectors in \( \mathbb{R}^4 \) whose entries sum to zero.

d) The line \( \{(t, 2t, 3t): t \in \mathbb{R}\} \).

e) \( \mathbb{R}^3 \)
Problem 3. [25 points]

In this problem we will consider the best-fit plane \( z(x, y) = Bx + Cy + D \) through the data points

\[
\begin{pmatrix}
3 \\
-5 \\
b_1 \\
b_2 \\
1 \\
1 \\
-1 \\
5 \\
b_3 \\
-1 \\
3 \\
-7 \\
b_4
\end{pmatrix}.
\]

a) Find the matrix \( A \) such that the coefficient vector \( \vec{x} = (B, C, D) \) is the least-squares solution of

\[
A \begin{pmatrix} B \\ C \\ D \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}.
\]

b) Compute the QR decomposition of \( A \).
c) Given a vector \( b = (b_1, b_2, b_3, b_4) \), explain how a computer would quickly compute \( \hat{x} \) using the QR decomposition you found in b).

d) Compute the best-fit plane \( z = Bx + Cy + D \) when \( (b_1, b_2, b_3, b_4) = (10, -10, 20, -20) \), using your QR decomposition or otherwise.

e) What is the minimum value of
\[
(z(3, -5) - 10)^2 + (z(1, 1) + 10)^2 + (z(-1, 5) - 20)^2 + (z(3, -7) + 20)^2
\]
for all planes \( z = Bx + Cy + D \)?
Problem 4. [20 points]

Consider the subspace $V$ in $\mathbb{R}^4$ defined by $x_1 + 2x_2 - 2x_3 - x_4 = 0$.

a) Compute the orthogonal projection $b_{V \perp}$ of the vector $b = (0, -3, 3, -2)$ onto $V^\perp$.

b) Compute the orthogonal decomposition $b = b_V + b_{V \perp}$.

c) Find the matrix $P_V$ for orthogonal projection onto $V$.

d) Find a basis for $\text{Nul}(P_V)$. 
Problem 5. [16 points]

a) Let $A$ be an $m \times n$ matrix of rank $r$. Which of the following statements are equivalent to “$A$ has full row rank”?

1) $\text{Nul}(A^T) = \{0\}$
2) $n = r$
3) $\text{Col}(A) = \mathbb{R}^m$
4) $A$ has linearly independent columns
5) $A$ has a pivot in every row
6) $A$ is invertible
7) $Ax = b$ is consistent for every vector $b$

b) Explain why the projection matrix $P_V$ onto a subspace $V$ can be written as $QQ^T$ for some matrix $Q$ with orthonormal columns. (What is $Q$ in terms of $V$?)

c) Find three nonzero vectors $v_1, v_2, v_3 \in \mathbb{R}^3$ such that $\{v_1, v_2, v_3\}$ is linearly dependent, but $v_3$ is not in $\text{Span}\{v_1, v_2\}$. Be sure to label which is $v_3$.

d) Give an example of a $4 \times 4$ matrix $A$ such that $\text{Nul}(A) = \text{Row}(A)$, or explain why no such matrix exists.
Problem 6. [10 points]

Unit vectors $u_1$ and $u_2$ and a vector $b$ are drawn in the picture below. In the same picture, draw and label a) $(b \cdot u_1)u_1$ and $(b \cdot u_2)u_2$ and b) $(b \cdot u_1)u_1 + (b \cdot u_2)u_2$.

c) Note that $\{u_1, u_2\}$ is a basis for $V = \mathbb{R}^2$. Explain why $b = b_v \neq (b \cdot u_1)u_1 + (b \cdot u_2)u_2$ does not contradict the projection formula.