Please read all instructions carefully before beginning.

- Do not open this test booklet until you are directed to do so.
- You have 75 minutes to complete this exam.
- If you finish early, go back and check your work.
- You may use a calculator for doing arithmetic, but you should not need one. All other materials and aids are strictly prohibited.
- For full credit you must show your work so that your reasoning is clear, unless otherwise indicated.
- Do not spend too much time on any one problem. Read them all through first and attack them in an order that allows you to make the most progress.
- Good luck!
Problem 1.  \[25 \text{ points}\]

Consider the matrix
\[ A = \begin{pmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{pmatrix}. \]

a) Use Gauss–Jordan elimination to put \( A \) into reduced row echelon form.

b) The free columns are \[ \square \].

c) The rank of \( A \) is \[ \square \].

d) Draw a picture of the column space \( \text{Col}(A) \) below.

![Matrix diagram]

e) Write down a vector \( b \) in \( \mathbb{R}^2 \) such that \( Ax = b \) has no solution. If no such vector exists, explain why not.

f) The null space is a (circle one) \( \text{point} \) \( \text{line} \) \( \text{plane} \) in (fill in the blank) \( \mathbb{R}^{\square} \).

g) Find the general solution of \( Ax = \begin{pmatrix} -2 \\ 4 \end{pmatrix} \) in parametric vector form.

h) Express \( \text{Nul}(A) \) as a span of some number of vectors.

i) Write down any nontrivial solution of \( Ax = 0 \).
Solution.

a) An REF is \[
\begin{pmatrix}
1 & -1 & 2 \\
0 & 0 & 0
\end{pmatrix}.
\]

b) The free columns are the second and third.

c) The rank is 1.

d) 

![Matrix Diagram]

e) Any \( b \) not on this line works. For instance, \( b = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \).

f) The null space is a plane in \( \mathbb{R}^3 \).

g) 
\[
x = \begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}
\]

h) 
\[
\text{Nul}(A) = \text{Span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \right\}
\]

i) For instance, \((1, 1, 0)\).
Problem 2. [25 points]

Consider the matrix

\[ A = \begin{pmatrix} 2 & 3 & 1 \\ -4 & -5 & -3 \\ -2 & -6 & 0 \end{pmatrix}. \]

a) Find a lower-unitriangular matrix \( L \) and a matrix \( U \) in REF such that \( A = LU \). You should end up with

\[ U = \begin{pmatrix} 2 & 3 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & -2 \end{pmatrix}. \]

b) Solve the equation \( Ax = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix} \) using the \( LU \) decomposition you found in a).

c) If you compute a \( PA = LU \) decomposition using maximal partial pivoting, what is \( P \)? (You do not have to do the bookkeeping to find \( L \) in this part.)

d) Find the inverse of \( A \).

Solution.

a) \[ L = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -3 & 1 \end{pmatrix} \]

b) \[ x = \begin{pmatrix} 4 \\ -2 \\ -2 \end{pmatrix} \]

c) \[ P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \]

d) \[ A^{-1} = \frac{1}{4} \begin{pmatrix} 18 & 6 & 4 \\ -6 & -2 & -2 \\ -14 & -6 & -2 \end{pmatrix} \]
Problem 3. [20 points]

Consider the subspace $V$ of $\mathbb{R}^4$ defined by the equations

\[\begin{align*}
x_1 + x_2 &= x_3 + x_4 \\
x_1 + x_3 &= x_2 + x_4.
\end{align*}\]

a) Express $V$ as the null space of a matrix $A$.

b) Express $V$ as the span of a set of vectors.

c) Express $V$ as the column space of a (different) matrix $B$.

d) One of the following vectors is contained in $V$. Identify which one is contained in $V$, and express it as a linear combination of the vectors you found in b).

\[
\begin{pmatrix}
1 \\
4 \\
2 \\
3
\end{pmatrix}
\begin{pmatrix}
1 \\
3 \\
3 \\
1
\end{pmatrix}
\begin{pmatrix}
1 \\
3 \\
-3 \\
-1
\end{pmatrix}
\]

Solution.

a) $V = \text{Nul} \begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix}$

b) $V = \text{Span} \left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

c) $V = \text{Col} \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$

d) \[
\begin{pmatrix} 1 \\ 3 \\ 3 \\ 1 \end{pmatrix} = 3 \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}\]
Problem 4. [15 points]

Let
\[ A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 5 & 6 & 7 & 8 \\ 6 & 8 & 10 & 12 \\ -9 & -10 & -11 & -12 \end{pmatrix}. \]

a) What three row operations are needed to transform \( A \) into \( B \)?

b) What are the elementary matrices \( E_1, E_2, E_3 \) for these three operations?

c) Write an equation for \( B \) in terms of \( A \) and \( E_1, E_2, E_3 \).

d) Write an equation for \( A \) in terms of \( B \) and \( E_1^{-1}, E_2^{-1}, E_3^{-1} \).

Solution.

a) First \( R_1 \leftrightarrow R_2 \), then \( R_2 += R_1 \), then \( R_3 \times = -1 \).

b) \( E_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \)

c) \( B = E_3 E_2 E_1 A \)

d) \( A = E_1^{-1} E_2^{-1} E_3^{-1} B \)
Problem 5. [10 points]

Consider the subspace

\[ V = \text{Span}\left\{ \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix}, \begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix} \right\}. \]

a) Show that \( \begin{pmatrix} -4 \\ -4 \\ -4 \end{pmatrix} \) is in \( V \).

b) Show that \( \begin{pmatrix} -4 \\ -4 \\ 4 \end{pmatrix} \) is not in \( V \).

c) Circle one: \( V \) is a point line plane space.

Solution.

a) We solve the vector equation

\[ x_1 \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix} + x_3 \begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix} = \begin{pmatrix} -4 \\ -4 \\ -4 \end{pmatrix} \]

by row reducing an augmented matrix:

\[
\begin{pmatrix}
1 & 2 & 3 & -4 \\
4 & 5 & 6 & -4 \\
7 & 8 & 9 & -4
\end{pmatrix}
\xrightarrow{\text{RREF}}
\begin{pmatrix}
1 & 0 & -1 & 4 \\
0 & 1 & 2 & -4 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

This system has infinitely many solutions, so \( (-4, -4, -4) \) is in \( V \).

b) We solve the vector equation

\[ x_1 \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix} + x_3 \begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix} = \begin{pmatrix} -4 \\ -4 \\ 4 \end{pmatrix} \]

by row reducing an augmented matrix:

\[
\begin{pmatrix}
1 & 2 & 3 & -4 \\
4 & 5 & 6 & -4 \\
7 & 8 & 9 & 4
\end{pmatrix}
\xrightarrow{\text{RREF}}
\begin{pmatrix}
1 & 0 & -1 & 0 \\
0 & 1 & 2 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

This system has no solutions, so \( (-4, -4, 4) \) is not in \( V \).

c) Since \( V \) contains two noncollinear vectors and is not all of \( \mathbb{R}^3 \), it must be a plane.
Problem 6. [15 points]

Find examples of matrices with the following properties. If no such matrix exists, explain why not.

a) A $2 \times 3$ matrix $A$ such that the solution set of $Ax = 0$ is the line spanned by $(1, 2, 1)$ and the solution set of $Ax = \binom{2}{1}$ is the point $\{(1, 0, 0)\}$.

b) A $3 \times 2$ matrix $A$ such that $\text{Col}(A) = \mathbb{R}^3$.

c) A $2 \times 3$ matrix $A$ such that $\text{Col}(A) = \mathbb{R}^3$.

d) A $2 \times 3$ matrix $A$ such that $\text{Nul}(A) = \mathbb{R}^3$.

e) An invertible $2 \times 2$ matrix such that $A\binom{1}{2} = A\binom{2}{1}$.

Solution.

a) Impossible: the solution set of $Ax = \binom{2}{1}$ is a translate of the solution set of $Ax = 0$.

b) Impossible: the two columns of $A$ cannot span anything larger than a plane.

c) Impossible: the column space of $A$ lives in $\mathbb{R}^2$.

d) The only example is the zero matrix.

e) Impossible: the equation $Ax = b$ has exactly one solution.