Please read all instructions carefully before beginning.

- Do not open this test booklet until you are directed to do so.
- You have 75 minutes to complete this exam.
- If you finish early, go back and check your work.
- You may use a calculator for doing arithmetic, but you should not need one. All other materials and aids are strictly prohibited.
- For full credit you must show your work so that your reasoning is clear, unless otherwise indicated.
- Do not spend too much time on any one problem. Read them all through first and attack them in an order that allows you to make the most progress.
- Good luck!

This is a practice exam. It is meant to be similar in format, length, and difficulty to the real exam. It is not meant as a comprehensive list of study problems. I recommend completing the practice exam in 75 minutes, without notes or distractions.
Problem 1. [20 points]

Consider

\[
A = \begin{pmatrix}
0 & 1 & -1 & 0 \\
-2 & -1 & -1 & 2 \\
2 & 3 & 5 & 4 \\
6 & 3 & -3 & 0 \\
\end{pmatrix} \quad b = \begin{pmatrix}
-2 \\
-8 \\
4 \\
0 \\
\end{pmatrix}.
\]

a) Carry out Gaussian reduction with maximal partial pivoting to find a \( PA = LU \) decomposition. You should obtain

\[
U = \begin{pmatrix}
6 & 3 & -3 & 0 \\
0 & 2 & 6 & 4 \\
0 & 0 & -4 & -2 \\
0 & 0 & 0 & 3 \\
\end{pmatrix}.
\]

Please write the row operations you performed.

\[
L = \begin{pmatrix}
\end{pmatrix} \quad P = \begin{pmatrix}
\end{pmatrix}
\]
b) Write the elementary matrices for the row operations you performed.

c) Solve the equations $Ly = Pb$ and $Ux = y$ to find a solution of $Ax = b$.

\[
\begin{pmatrix}
    x_1 \\
    x_2 \\
    \vdots \\
\end{pmatrix}
\]

d) Briefly explain why step b) is faster than solving $Ax = b$ using Gaussian elimination on the augmented matrix $(A | b)$, once you have a $PA = LU$ decomposition.
Problem 2. [15 points]

a) Compute the inverse of \[
\begin{pmatrix}
1 & -2 & 3 \\
-2 & 6 & -5 \\
2 & 3 & 9
\end{pmatrix}.
\]
Be sure to write out any row operations you perform.

\[
\begin{pmatrix}
1 & -2 & 3 \\
-2 & 6 & -5 \\
2 & 3 & 9
\end{pmatrix}^{-1} = \begin{pmatrix}
\end{pmatrix}
\]

b) For which value(s) of \(k\) is \[
\begin{pmatrix}
1 & -2 & 3 \\
-2 & 6 & k \\
2 & 3 & 9
\end{pmatrix}
\]
not invertible?

\[
k = \begin{pmatrix}
\end{pmatrix}
\]

c) Suppose that \(A\) is a \(3 \times 3\) matrix whose third column is in the span of the first two. Briefly explain why \(A\) is not invertible.
Problem 3.  

Consider

\[
A = \begin{pmatrix}
1 & 3 & -2 & 0 \\
-2 & -6 & 6 & -2 \\
2 & 6 & 3 & -7 \\
\end{pmatrix} \quad b = \begin{pmatrix}
2 \\
-8 \\
-10 \\
\end{pmatrix}.
\]

a) Find the parametric vector form of the solution set of \( Ax = b \). Be sure to write out any row operations you perform.

\[
x = \begin{pmatrix}
\quad \\
\quad \\
\quad \\
\end{pmatrix} + \begin{pmatrix}
\quad \\
\quad \\
\quad \\
\end{pmatrix}
\]

b) Write down two different solutions of \( Ax = b \). (Your answer will be two vectors with numbers in them.)

\[
x_1 = \begin{pmatrix}
\quad \\
\quad \\
\quad \\
\end{pmatrix} \quad x_2 = \begin{pmatrix}
\quad \\
\quad \\
\quad \\
\end{pmatrix}
\]

c) Find a spanning set for \( \text{Nul}(A) \).

\[\text{Nul}(A) = \text{Span}\left\{ \begin{pmatrix}
\quad \\
\quad \\
\quad \\
\end{pmatrix}, \begin{pmatrix}
\quad \\
\quad \\
\quad \\
\end{pmatrix}, \begin{pmatrix}
\quad \\
\quad \\
\quad \\
\end{pmatrix} \right\}\]

d) Let \( v = (-1, 1, 1, 1) \). Check that \( v \in \text{Nul}(A) \), and write \( v \) as a linear combination of the spanning vectors you obtained in d).

[Hint: what values do the free variables have to take?]
Problem 4.  

For a certain $2 \times 2$ matrix $A$, the solution set of $Ax = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is drawn.

a) Draw the solution set of $Ax = 0$ and the solution set of $Ax = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$ in the grid below. Be sure to label which is which.

b) $\operatorname{rank}(A) =$

c) Draw the column space of $A$. Be precise!
Problem 5. [15 points]

Find examples of matrices with the following properties. If no such matrix exists, write "no way, man," or use your favorite colloquialism instead. You need not justify your answers.

a) A matrix $A$, in RREF, such that $Ax = b$ has at least one solution for every $b$, but $A$ does not have a pivot in every column.

b) A $3 \times 5$ matrix of rank 4, in RREF.

c) A $2 \times 2$ matrix $A$ such that the solution set of $Ax = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ is a line, and $Ax = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ has no solutions.

d) A $3 \times 3$ matrix $A$ with no zero entries, such that $\text{Col}(A)$ is a plane.

e) A $4 \times 4$ matrix $A$ with a pivot in every row such that $A(1, 2, -1, 1) = 0$. 
Problem 6.  

Which of the following are subspaces of $\mathbb{R}^4$? If not, why?

a) $\text{Span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ -1 \\ -1 \\ 2 \end{pmatrix} \right\}$

b) $\text{Nul} \begin{pmatrix} 2 & -1 & 3 \\ 0 & 0 & 4 \\ 6 & -4 & 2 \\ -9 & 3 & 4 \end{pmatrix}$

c) $\text{Col} \begin{pmatrix} 2 & -1 & 3 \\ 0 & 0 & 4 \\ 6 & -4 & 2 \\ -9 & 3 & 4 \end{pmatrix}$

d) $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

e) $\{\}$

f) $V = \left\{ \text{all vectors} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \text{ in } \mathbb{R}^4 \text{ such that } xy = zw \right\}$