MATH 218D-1
PRACTICE FINAL EXAMINATION

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Please **read all instructions** carefully before beginning.

- Do not open this test booklet until you are directed to do so.
- You have 180 minutes to complete this exam.
- If you finish early, go back and check your work.
- You may use a **calculator** for doing arithmetic, but you should not need one. You may use a 8.5×11" **note sheet** as well. All other materials and aids are strictly prohibited.
- For full credit you must **show your work** so that your reasoning is clear, unless otherwise indicated.
- Do not spend too much time on any one problem. Read them all through first and attack them in an order that allows you to make the most progress.
- Good luck!

This is a practice exam. It is meant to be similar in format, length, and difficulty to the real exam. It is **not** meant as a comprehensive list of study problems. I recommend completing the practice exam in 180 minutes, without notes or distractions.

Problem 1. [20 points]

Consider the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 3 & 1 \end{pmatrix}.$$

a) Compute the $A = LU$ decomposition of A.

b) Compute A^{-1}.

c) Express A^{-1} as a product of elementary matrices.

d) Solve $Ax = \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}$.

Solution.

a)

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 3 & 1 \end{pmatrix} \qquad U = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

b)

$$A^{-1} = \begin{pmatrix} -1 & 2 & -1 \\ -1 & 1 & 0 \\ 3 & -3 & 1 \end{pmatrix}$$

c)

$$A^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

d)

$$x = \begin{pmatrix} -2 \\ -3 \\ 8 \end{pmatrix}$$

Problem 2. [20 points]

A certain 3×2 matrix A satisfies $AA^T = QDQ^T$ for

$$Q = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ -1/\sqrt{3} & 0 & 2/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \end{pmatrix} \qquad D = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 1/9 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

a) What is the rank of A?

b) What are the singular values of A?

c) What are the maximum and minimum values of $\|Ax\|$ subject to $\|x\| = 1$?

d) Find orthonormal bases for Col(A) and Nul(A^T).

Suppose now that the columns of A are *orthogonal*.

e) What are the two possibilities for $A^T A$?

f) You have enough information to determine the columns of A up to sign: the longer column is $\pm(?)$, and the shorter column is $\pm(?)$.

Solution.

a) The rank is 2.

b) The singular values are 2 and $1/3$.

c) The maximum value is 2, and the minimum value is $1/3$.

d) $$\text{Col}(A): \left\{ \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\} \qquad \text{Nul}(A^T): \left\{ \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right\}$$

e) If the columns are orthogonal then $A^T A = \begin{pmatrix} 4 & 0 \\ 0 & 1/9 \end{pmatrix}$ or $\begin{pmatrix} 1/9 & 0 \\ 0 & 4 \end{pmatrix}$.

f) Let u_1, u_2 be the first two columns of Q. Then $\pm u_1$ is the longer column divided by $\sigma_1 = 2$, so the longer column is $\pm\frac{2}{\sqrt{3}}(1, -1, 1)$. Likewise, $\pm u_2$ is the shorter column divided by $\sigma_2 = 1/3$, so the shorter column is $\pm\frac{1}{3\sqrt{2}}(1, 0, -1)$.

Problem 3. [20 points]

Consider the three data points

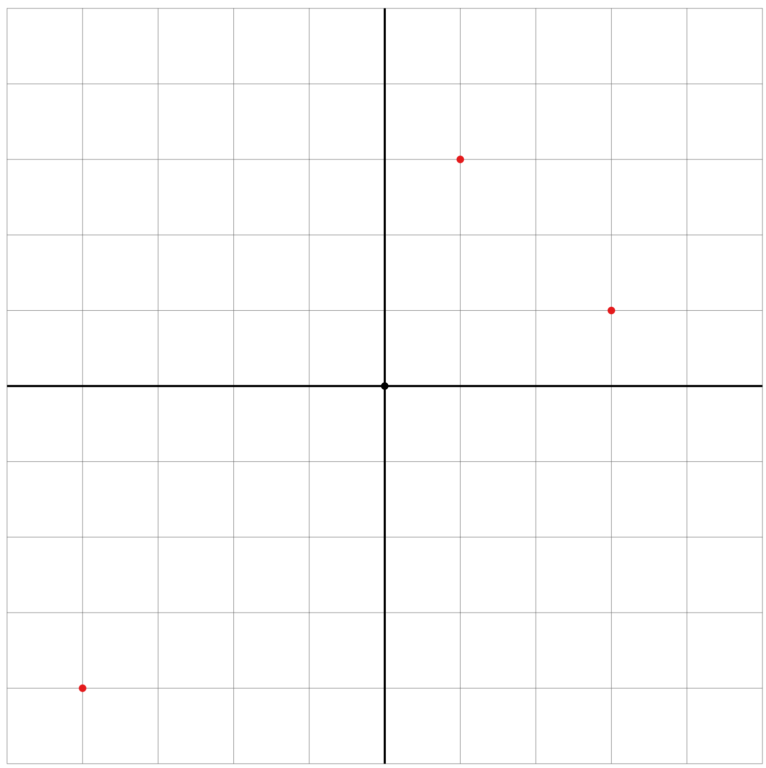
$$\begin{pmatrix} 3 \\ 1 \end{pmatrix}, \begin{pmatrix} -4 \\ -4 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

a) Form the matrix A with the data points as columns, and note that the row averages are equal to zero.

b) Compute the covariance matrix $S = \frac{1}{3-1}AA^T$. What are its eigenvalues $\lambda_1 > \lambda_2$ and corresponding unit eigenvectors v_1, v_2?

c) Sketch the line in the direction of largest variance in the grid below.

d) What is the variance of the data in the direction perpendicular to the line you drew in **c)**?



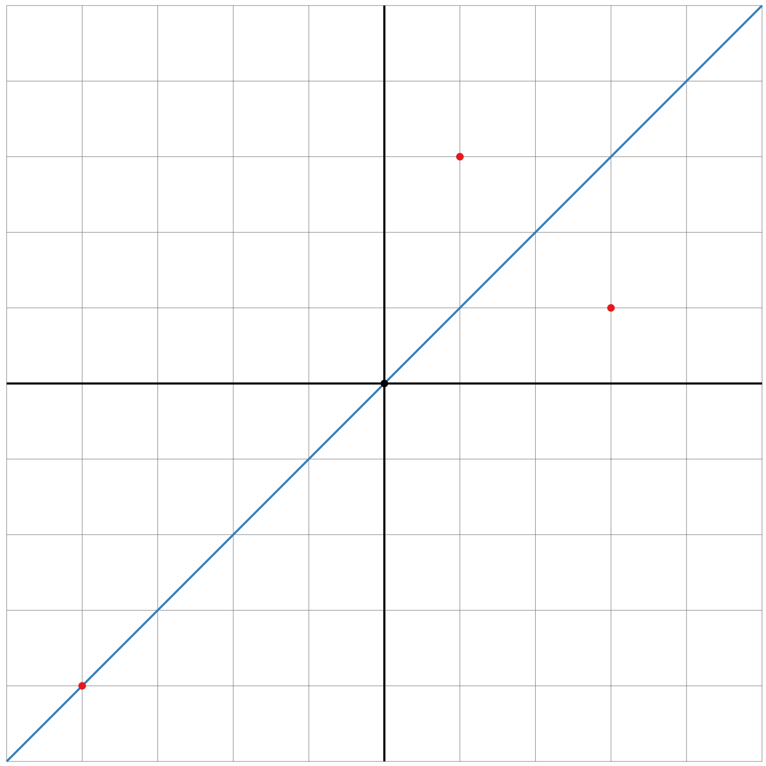
The data points in the problem.

Solution.

a)
$$A = \begin{pmatrix} 3 & -4 & 1 \\ 1 & -4 & 3 \end{pmatrix}$$

b) $S = \begin{pmatrix} 13 & 11 \\ 11 & 13 \end{pmatrix}$ $\quad \lambda_1 = 24$ $\quad \lambda_2 = 2$ $\quad v_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\quad v_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

c)



d) The variance in the perpendicular direction is $\sigma_2^2 = \lambda_2 = 2$.

Problem 4.

[20 points]

Consider the three data points

$$\begin{pmatrix}3\\1\end{pmatrix}, \begin{pmatrix}-4\\-4\end{pmatrix}, \begin{pmatrix}1\\3\end{pmatrix}$$

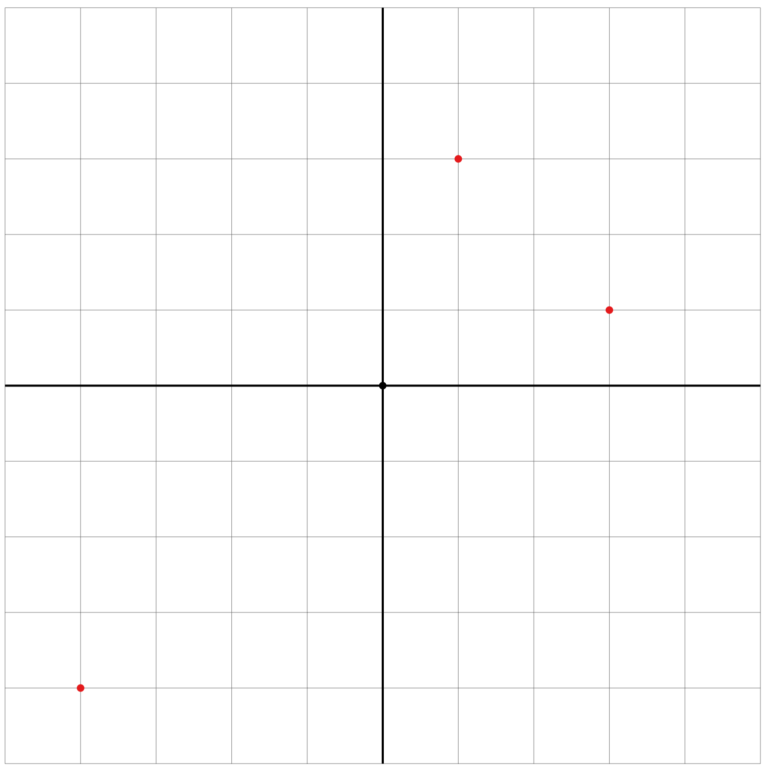
of the previous problem. In this problem, we will approximate these data points with a best-fit line in the least-squares sense.

a) The general equation of a line in the plane is $y = Bx + C$. Find a system of linear equations that would be satisfied by a line passing through the above three points, and write this as a matrix equation $Ax = b$.

b) Find the least-squares solution $\widehat{x}$ of $Ax = b$. What is the equation of the best-fit line?

c) Sketch the best-fit line in the grid below.

d) The line you drew in this problem is different than the line you drew in the previous problem. How can that be?



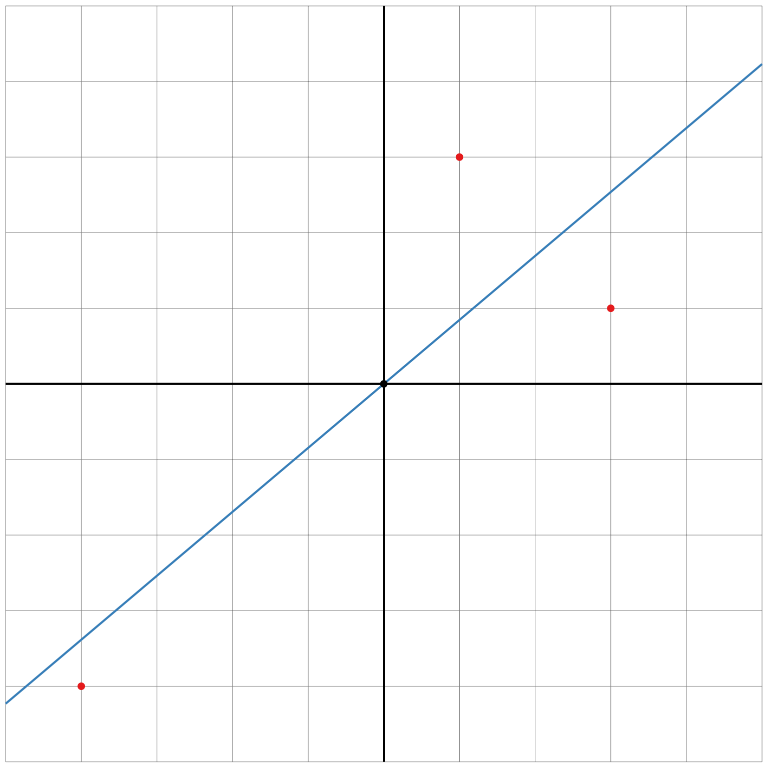
The data points in the problem.

Solution.

a) $$\begin{aligned} 1 &= 3B + C \\ -4 &= -4B + C \\ 3 &= B + C \end{aligned} \quad \rightsquigarrow \quad \begin{pmatrix} 3 & 1 \\ -4 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} B \\ C \end{pmatrix} = \begin{pmatrix} 1 \\ -4 \\ 3 \end{pmatrix}$$

b) $$A^T A = \begin{pmatrix} 26 & 0 \\ 0 & 3 \end{pmatrix} \qquad A^T b = \begin{pmatrix} 22 \\ 0 \end{pmatrix} \quad \rightsquigarrow \quad \widehat{x} = \begin{pmatrix} 11/13 \\ 0 \end{pmatrix} \quad \rightsquigarrow \quad y = \frac{11}{13} x$$

c)



d) The least-squares best-fit line minimizes *vertical* distance from the data points to the line; the PCA best-fit line minimizes the *orthogonal* distance.

Problem 5. [15 points]

Consider the matrix

$$A = \begin{pmatrix} 4 & 2 & -4 \\ 0 & 2 & 0 \\ 2 & 2 & -2 \end{pmatrix}.$$

a) Find an invertible matrix C and a diagonal matrix D such that $A = CDC^{-1}$.

b) Compute $A^{100}\begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}$.

Solution.

a)
$$C = \begin{pmatrix} -1 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

b)
$$\begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad \rightsquigarrow \quad A^{100}\begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix} = 2^{100}\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

Problem 6. [25 points]

Consider the subspace

$$W = \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 4 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \\ 4 \\ -1 \end{pmatrix}, \begin{pmatrix} 4 \\ -8 \\ -2 \\ 0 \end{pmatrix} \right\}.$$

a) Find an orthonormal basis of W.

Now consider the subspace V with orthonormal basis

$$\left\{ \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{10}}\begin{pmatrix} 1 \\ -2 \\ -1 \\ 2 \end{pmatrix}, \frac{1}{\sqrt{10}}\begin{pmatrix} 2 \\ 1 \\ -2 \\ -1 \end{pmatrix} \right\}$$

(The reason for changing subspaces is to prevent carry-through error in the rest of the problem.)

b) Compute the matrix P_V for orthogonal projection onto V. (You can write P_V as a product of two matrices without multiplying it out.)

c) Find the orthogonal decomposition $x = x_V + x_{V^\perp}$ for $x = (1,0,1,0)$. Your answer should involve fractions and not decimals.

d) Find a basis for $V^\perp$.

[**Hint:** you already did this in **c)**.]

e) Find an implicit equation for V: that is, $V = \big\{(x_1, x_2, x_3, x_4) : (?) = 0\big\}$.

f) Orthogonally diagonalize P_V: that is, find an orthogonal matrix Q and a diagonal matrix D such that $P_V = QDQ^T$.

[**Hint:** You have already done all of the necessary calculations.]

Solution.

a)
$$\left\{ \frac{1}{\sqrt{6}}\begin{pmatrix}1\\-1\\0\\-2\end{pmatrix}, \frac{1}{\sqrt{21}}\begin{pmatrix}2\\0\\4\\1\end{pmatrix}, \frac{1}{\sqrt{15}}\begin{pmatrix}1\\-3\\-1\\2\end{pmatrix} \right\}$$

b)
$$P_V = QQ^T \quad \text{for} \quad Q = \begin{pmatrix}1/2 & 1/\sqrt{10} & 2/\sqrt{10}\\ 1/2 & -2/\sqrt{10} & 1/\sqrt{10}\\ 1/2 & -1/\sqrt{10} & -2/\sqrt{10}\\ 1/2 & 2/\sqrt{10} & -1/\sqrt{10}\end{pmatrix}$$

c)
$$x = \frac{1}{2}\begin{pmatrix}1\\1\\1\\1\end{pmatrix} + \frac{1}{2}\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix}$$

d)
$$V^{\perp} = \text{Span}\left\{\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix}\right\}$$

e)
$$V = \{(x_1, x_2, x_3, x_4) \colon x_1 - x_2 + x_3 - x_4 = 0\}$$

f)
$$Q = \begin{pmatrix}1/2 & 1/\sqrt{10} & 2/\sqrt{10} & 1/2\\ 1/2 & -2/\sqrt{10} & 1/\sqrt{10} & -1/2\\ 1/2 & -1/\sqrt{10} & -2/\sqrt{10} & 1/2\\ 1/2 & 2/\sqrt{10} & -1/\sqrt{10} & -1/2\end{pmatrix} \qquad D = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}$$

Problem 7. [20 points]

True/false questions: no justification is needed.

a) **T** **F** If $Ax = b$ has at least one solution for every $b \in \mathbf{R}^m$, then A has full row rank.

b) **T** **F** Every elementary matrix is invertible.

c) **T** **F** If x is in a subspace V, then the projection of x onto V is the zero vector.

d) **T** **F** A triangular matrix A with real entries can have a complex (non-real) eigenvalue.

e) **T** **F** A diagonalizable $n \times n$ matrix admits n linearly independent eigenvectors.

f) **T** **F** The maximum value of $||Ax||$ subject to $||x|| = 1$, is the largest eigenvalue of A.

g) **T** **F** If A is an $m \times n$ matrix with linearly dependent columns, then the columns of A do not span $\mathbf{R}^m$.

h) **T** **F** If A and B are $n \times n$ matrices and $\det(A) = 0$, then the columns of AB are linearly dependent.

i) **T** **F** If A has linearly independent columns, then A^+A is the identity matrix.

j) **T** **F** If λ is a eigenvalue of A^TA and $\lambda > 0$, then λ is also an eigenvalue of AA^T.

Solution.

a) True.

b) True.

c) False: it is x.

d) False: the eigenvalues are the diagonal entries, which are real.

e) True.

f) False: it is the largest singular value.

g) False: a matrix can have full row rank but not full column rank.

h) True: $\det(AB) = \det(A)\det(B) = 0$, so AB is not invertible.

i) True: A^+A is the projection onto the row space, which is all of $\mathbf{R}^n$ because $\mathrm{Nul}(A) = \{0\}$.

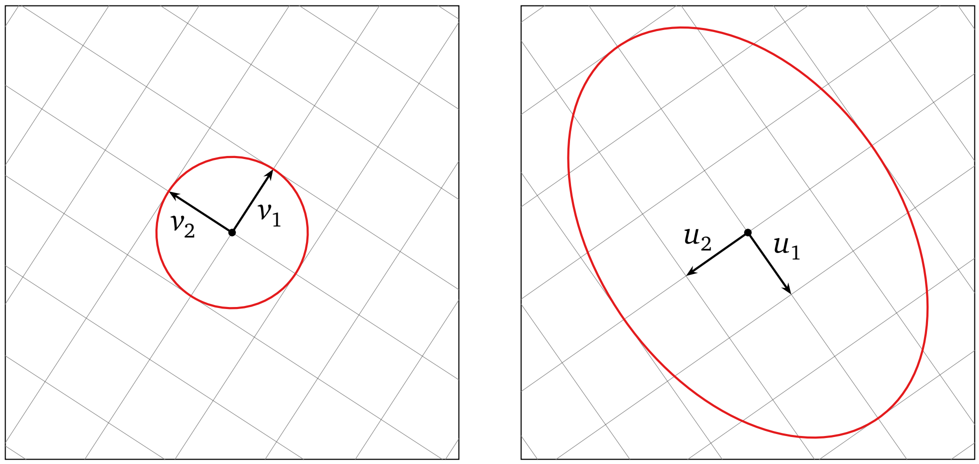
j) True.

Problem 8. [10 points]

A certain 2×2 matrix A has the singular value decomposition

$$A = 3u_1 v_1^T + 2u_2 v_2^T$$

where u_1, u_2, v_1, v_2 are drawn in the diagrams below. The unit circle $\{x : \|x\| = 1\}$ is drawn on the left. Draw $\{Ax : \|x\| = 1\}$ on the right.



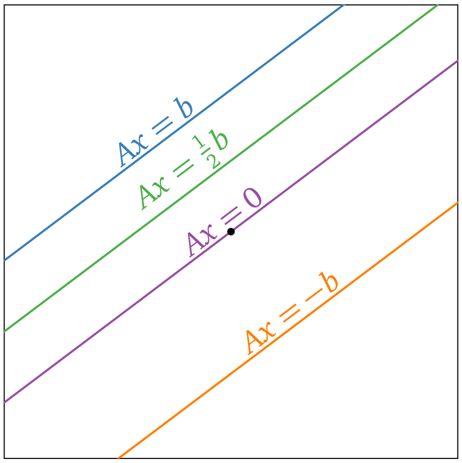
Problem 9. [15 points]

The solution set of $Ax = b$ for a certain 3×2 matrix A is drawn below.

a) *Draw and label* the solution sets of $Ax = \frac{1}{2}b$, $Ax = 0$, and $Ax = -b$ on the same diagram.

b) What is the rank of A?

c) Explain why there exists some vector $b' \in \mathbf{R}^3$ such that $Ax = b'$ is inconsistent.



Solution.

b) The rank is 1.

c) Because A does not have full row rank.