1. **Projection onto a line**
   For each of the following,
   (1) project the vector $b$ onto the line $V = \text{Span}\{v\}$;
   (2) draw the three vectors $b, b_v, b_{v \perp}$;
   (3) compute the projection matrix $P = \frac{vv^T}{v^Tv}$.
   
   a) $b = (1, 1), \ v = (1, 0)$
   
   b) $b = (0, 2), \ v = (1, 1)$
   
   c) $b = (1, 2, 3), \ v = (1, 1, -1)$. 
2. **Planes and normal vectors**

   The subspace \( V = \text{Span}\{(1, 1, 2), (1, 3, 1)\} \) of \( \mathbb{R}^3 \) is a plane.

   **a)** Make the vectors \((1, 1, 2), (1, 3, 1)\) into the rows of a \(2 \times 3\) matrix \( A\) - this means that \( \text{Row}(A) = V\). Find a basis for \( \text{Nul}(A)\). Since

   \[
   V^\perp = \text{Row}(A)^\perp = \text{Nul}(A),
   \]

   you have found a basis \( v = (a, b, c) \) for the line \( V^\perp\).

   In other words, you have found a basis for \( V^\perp \) by solving the two orthogonality equations

   \[
   (a, b, c) \cdot (1, 1, 2) = a + b + 2c = 0,
   \]

   \[
   (a, b, c) \cdot (1, 3, 1) = a + 3b + c = 0.
   \]

   **b)** Confirm that \( V \) is the plane \( \{(x, y, z) \in \mathbb{R}^3 : ax + by + cz = 0\} \), by showing that both \((1, 1, 2)\) and \((1, 3, 1)\) solve this equation. **The coefficients of a plane’s equation make a normal vector for the plane.**

   **c)** Find the orthogonal decomposition \( b = b_v + b_{V^\perp} \) of the vector \( b = (1, 1, 1) \) with respect to the plane \( V \) and the orthogonal line \( V^\perp \).

   **Hint:** It is easier to compute \( b_{V^\perp} \), as it is the projection of \( b \) onto the line \( V^\perp \) spanned by the vector \( v = (a, b, c) \).
3. Projection onto a plane

Consider the plane

\[ V = \text{Span}\{(1, 1, 1, 1), (1, 2, 3, 4)\} \]

in \( \mathbb{R}^4 \). We will find the orthogonal projection of \( b = (1, -1, -3, -5) \) onto \( V \). This is a vector \( b_V \in \mathbb{R}^4 \) so that \( b_V \in V \) and \( b_V \perp = b - b_V \in V^\perp \).

Since \( b_V \) is in \( V \), it must equal

\[ b_V = \hat{x}_1(1, 1, 1, 1) + \hat{x}_2(1, 2, 3, 4) \]

for some scalars \( \hat{x}_1 \) and \( \hat{x}_2 \). We will compute the orthogonal projection by solving for these scalars.

The vector \( b_V \perp \) is orthogonal to every vector in \( V \), in particular it is orthogonal to both \( (1, 1, 1, 1) \) and \( (1, 2, 3, 4) \). We get two equations:

\[
\begin{align*}
(1, 1, 1, 1) \cdot b_V \perp &= 0, \\
(1, 2, 3, 4) \cdot b_V \perp &= 0.
\end{align*}
\]

Expanding \( b_V \perp = b - b_V = (1, -1, -3, -5) - (\hat{x}_1(1, 1, 1, 1) + \hat{x}_2(1, 2, 3, 4)) \), we can rewrite these two equations as

\[
\begin{align*}
(1, 1, 1, 1) \cdot (\hat{x}_1(1, 1, 1, 1) + \hat{x}_2(1, 2, 3, 4)) &= (1, 1, 1, 1) \cdot (1, -1, -3, -5), \\
(1, 2, 3, 4) \cdot (\hat{x}_1(1, 1, 1, 1) + \hat{x}_2(1, 2, 3, 4)) &= (1, 2, 3, 4) \cdot (1, -1, -3, -5).
\end{align*}
\]

a) By computing the dot-products, convert this into two linear equations in the two unknowns \( \hat{x}_1 \) and \( \hat{x}_2 \).

b) Solve for \( \hat{x}_1 \) and \( \hat{x}_2 \), and compute the orthogonal projection

\[ b_V = \hat{x}_1(1, 1, 1, 1) + \hat{x}_2(1, 2, 3, 4). \]

c) Confirm that the vector \( b_V \perp = b - b_V \) is orthogonal to \( V \) by checking that

\[ b_V \perp \cdot (1, 1, 1, 1) = 0 \text{ and } b_V \perp \cdot (1, 2, 3, 4) = 0. \]

d) Write down a matrix \( A \) whose column are the two vectors which span \( V \), and compute \( A^T A \), the “matrix of dot products”. Compute the vector \( A^T b \). Explain where the matrix equation \( A^T A \hat{x} = A^T b \) (the normal equation) appears in a)-b), and also where the product \( b_V = A \hat{x} \) appears.

e) Compute the projection matrix \( P = A(A^T A)^{-1}A^T \) for the subspace \( V \).

f) Compute the vectors \( (I_4 - P) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \) and \( (I_4 - P) \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \). Explain why these two vectors give a basis for the plane \( V^\perp \).

g) Use your answer to f) to describe the plane \( V \) via two implicit equations:

\[ V = \{ (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : c_1 x_1 + c_2 x_2 + c_3 x_3 + c_4 x_4 = 0 \text{ and } c'_1 x_1 + c'_2 x_2 + c'_3 x_3 + c'_4 x_4 = 0 \}. \]

In other words, what coefficient vectors \( (c_1, c_2, c_3, c_4) \) and \( (c'_1, c'_2, c'_3, c'_4) \) can we use to describe \( V \), and why? Confirm that every vector in \( V \) satisfies these equations by checking that both \( (1, 1, 1, 1) \) and \( (1, 2, 3, 4) \) do.
4. Some mistakes to avoid

A false “fact”: every projection matrix \( P = A(A^T A)^{-1} A^T \) equals the identity matrix \( I \).

A false “proof”:
\[
P = A(A^T A)^{-1} A^T = AA^{-1}(A^T)^{-1} A^T = (AA^{-1})(A^T)^{-1} A^T = I \cdot I = I.
\]

a) What is wrong with this proof?

b) In what case would this proof be correct?

Consider the subspace \( V = \text{Span}\{(1, 1, 1, -1), (2, 1, 1, 2), (3, 2, 2, 1)\} \) in \( \mathbb{R}^4 \). \( V \) is the column space of the matrix
\[
A = \begin{pmatrix}
1 & 2 & 3 \\
1 & 1 & 2 \\
1 & 1 & 2 \\
-1 & 2 & 1
\end{pmatrix}.
\]

c) It would be incorrect to say that \( P = A(A^T A)^{-1} A^T \) is the projection matrix for \( V \). Why?

Hint: Try computing \( P \) - what goes wrong?

d) How could you modify \( A \) so that \( P = A(A^T A)^{-1} A^T \) is the projection matrix for \( V \)?