1. Linear (in)dependence

a) Are the vectors \((\frac{1}{0}), (\frac{1}{1})\) linearly independent? If not, write down a linear dependence relation.

b) Are the vectors \((\frac{1}{1}), (\frac{1}{2})\) linearly independent? If not, write down a linear dependence relation.

c) What is the dimension of Span \(\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}?\) Why?

d) Consider 2 linearly independent vectors \(u, v \in \mathbb{R}^n\). Show that the two vectors \(u + v, u - v\) are linearly independent.

e) Consider 3 vectors \(u, v, w \in \mathbb{R}^n\). Show that the three vectors \(u + v, u + 2v - w, v - w\) are linearly dependent.

f) Show that the vectors
\[
\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -2 \\ -3 \\ -4 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}
\]

are linearly dependent, by writing down a linear dependence relation among them.

Hint: Write down the matrix \(A\) whose columns are these vectors, and find a non-zero vector in \(\text{Nul}(A)\). Why does this solve the question?
2. Bases from an LU decomposition
Suppose that you have an $A = LU$ decomposition, where

\[
U = \begin{pmatrix}
1 & -1 & 2 & 3 & 5 \\
0 & 0 & 1 & 2 & 2 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix},
\]

but you don’t know $L$ or $A$.

a) Which of the subspaces $\text{Row}(A)$, $\text{Col}(A)$, $\text{Nul}(A)$, $\text{Nul}(A^T)$ can you find a basis for? If you can find a basis, do.

b) Which of the subspaces $\text{Row}(A)$, $\text{Col}(A)$, $\text{Nul}(A)$, $\text{Nul}(A^T)$ can you find the dimension of? If you can find a dimension, do.
3. Computing all of the bases at once

We will find a basis for each of the four fundamental subspaces of the $7 \times 10$ matrix

$$A = \begin{pmatrix} -2 & -2 & 1 & 2 & -4 & -4 & 1 & -8 & 9 & 1 \\ -4 & -3 & 0 & 0 & 11 & -5 & -9 & 5 & -5 & 3 \\ -5 & -5 & 1 & 3 & -1 & -8 & -3 & -10 & 8 & 6 \\ 3 & 3 & -1 & -2 & 2 & 5 & 0 & 9 & -9 & -4 \\ 4 & 4 & 0 & 1 & -11 & 4 & 8 & -4 & 7 & -10 \\ 2 & 2 & 0 & -3 & 5 & 4 & 1 & 6 & -3 & 1 \\ 3 & 3 & 0 & -3 & 3 & 6 & 2 & 8 & -5 & -1 \end{pmatrix}$$

by having a computer do Gauss–Jordan elimination one time. To do this, we will compute the RREF of the matrix $(A | I)$. The RREF is of the form $(U | E)$, and $EA = U$.

Load up linalg.js in your browser, and open a Javascript console. Copy this code to initialize the matrix:

```javascript
A = mat(
[-2,-2, 1, 2, -4,-4, 1, -8, 9, 1],
[-4,-3, 0, 0, 11,-5,-9, 5,-5, 3],
[-5,-5, 1, 3, -1,-8,-3,-10, 8, 6],
[ 3, 3,-1,-2, 2, 5, 0, 9,-9, -4],
[ 4, 4, 0, 1,-11, 4, 8, -4, 7,-10],
[ 2, 2, 0,-3, 5, 4, 1, 6,-3, 1],
[ 3, 3, 0,-3, 3, 6, 2, 8,-5, -1])
```

Now augment by the identity:

```javascript
// Create a blank 7x(10+7) matrix
Aug = Matrix.zero(7, 10+7);
// Put A in the left half
Aug.insertSubmatrix(0, 0, A);
// Augment with 7x7 identity matrix
Aug.insertSubmatrix(0, 10, Matrix.identity(7));
// Check that you got the right thing
console.log(Aug.toString(0));
```

Finally, compute the RREF:

```javascript
console.log(Aug.rref().toString(1));
```

a) The non-zero rows of $U$ form a basis for $\text{Row}(A)$. What are they?

b) The columns of $U$ with pivots are not a basis of $\text{Col}(A)$. However, the corresponding columns of $A$ do form a basis for $\text{Col}(A)$. What is a basis of $\text{Col}(A)$?

c) You can use $U$ (the RREF of $A$) to find a basis of $\text{Nul}(A) = \text{Nul}(U)$. What is a basis of $\text{Nul}(A)$?

d) The bottom row of $U$ is all zero. Look at the bottom row of $E$. This row forms a basis of $\text{Nul}(A^T)$ — what is it?
4. **Full row/column rank**

Let $A$ be an $m \times n$ matrix. Which of the following are equivalent to the statement “$A$ has full column rank”?

a) $\text{Nul}(A) = \{0\}$

b) $A$ has rank $m$

c) The columns of $A$ are linearly independent

d) $\dim \text{Row}(A) = n$

e) The columns of $A$ span $\mathbb{R}^m$

f) $A^T$ has full column rank

Which of the following are equivalent to the statement “$A$ has full row rank”?

a) $\text{Col}(A) = \mathbb{R}^m$

b) $A$ has rank $m$

c) The columns of $A$ are linearly independent

d) $\dim \text{Nul}(A) = n - m$

e) The rows of $A$ span $\mathbb{R}^n$

f) $A^T$ has full column rank