1. **Row Echelon Form**
   a) In REF, pivots are 1 and 3
   b) In REF, pivots are 5 and 6
   c) Not in REF
   d) In REF, pivots are 2 and 5
   e) Not in REF
   f) Not in REF

2. **Two Equations and Two Unknowns**
   a) Sub illustrate
   
   ![Graph showing two lines intersecting]

   b) The linear system is
      
      \[
      \begin{align*}
      x - y &= 2 \\
      2x - 4y &= -4.
      \end{align*}
      \]
      
      Subtract \(2 \cdot R_1\) from \(R_2\) to obtain:
      
      \[
      \begin{align*}
      x - y &= 2 \\
      -2y &= -8.
      \end{align*}
      \]
      
   c) Divide the second row by 2 to obtain:
      
      \[
      \begin{align*}
      x - y &= 2 \\
      y &= 4.
      \end{align*}
      \]
      
   d) Add the second row to the first row to obtain:
      
      \[
      \begin{align*}
      x &= 6 \\
      y &= 4.
      \end{align*}
      \]
      
      This is the solution.
e) $6 - 4 = 2$, $2 \cdot 6 - 4 \cdot 4 = -5$.

f) The system first becomes in REF after the 1st row operation. The pivots are 1 and $-2$. 
3. Three Equations Three Unknowns

a) \[ A = \begin{pmatrix} 1 & -3 & 1 \\ 2 & -8 & 8 \\ -6 & 3 & -15 \end{pmatrix}, \quad b = \begin{pmatrix} 4 \\ -2 \\ 9 \end{pmatrix}. \]

b) The augmented matrix is

\[
\begin{pmatrix}
1 & -3 & 1 & | & 4 \\
2 & -8 & 8 & | & -2 \\
-6 & 3 & -15 & | & 9
\end{pmatrix}.
\]

c) First, replace \( R_2 \) by \( R_2 - 2R_1 \) (\( R_2 \leftrightarrow -2R_1 \)).

Then \( R_3 \leftrightarrow 6R_1 \):

\[
\begin{pmatrix}
1 & -3 & 1 & | & 4 \\
0 & -2 & 6 & | & -10 \\
0 & -15 & -9 & | & 33
\end{pmatrix}.
\]

Now, you can do row scaling here, although you don’t need to. Let’s do it now to simplify our rows: \( R_2 \leftrightarrow -(1/2) \) and \( R_3 \leftrightarrow -(1/3) \) (combining two elementary operations at once):

\[
\begin{pmatrix}
1 & -3 & 1 & | & 4 \\
0 & 1 & -3 & | & 5 \\
0 & 5 & 3 & | & -11
\end{pmatrix}.
\]

We do one more row addition, replacing \( R_2 \) with \( R_2 - 5R_1 \) (\( R_2 \leftrightarrow -5R_1 \)).

\[
\begin{pmatrix}
1 & -3 & 1 & | & 4 \\
0 & 1 & -3 & | & 5 \\
0 & 0 & 18 & | & -36
\end{pmatrix}.
\]

Do one more row scaling, replacing \( R_3 \) with \( \frac{1}{18}R_3 \) (\( R_3 \leftrightarrow 1/18 \)).

\[
\begin{pmatrix}
1 & -3 & 1 & | & 4 \\
0 & 1 & -3 & | & 5 \\
0 & 0 & 1 & | & -2
\end{pmatrix}.
\]

d) I used 6 elementary row operations, but the row scalings could have been avoided, giving you as few as 3.

e) The system of equations is now

\[
\begin{align*}
x_1 & - 3x_2 + x_3 = 4 \\
x_2 & - 3x_3 = 5 \\
x_3 & = -2.
\end{align*}
\]
Substituting \( x_3 = -2 \), we obtain the system
\[
\begin{align*}
x_1 - 3x_2 &= 6 \\
x_2 &= -1 \\
x_3 &= -2.
\end{align*}
\]
Substituting \( x_2 = -1 \), we obtain the system
\[
\begin{align*}
x_1 &= 3 \\
x_2 &= -1 \\
x_3 &= -2,
\end{align*}
\]
which is the solution.

f) Check \[
\begin{pmatrix}
1 & -3 & 1 \\
2 & -8 & 8 \\
-6 & 3 & -15
\end{pmatrix}
\begin{pmatrix}
3 \\
-1 \\
2
\end{pmatrix}
= \begin{pmatrix}
4 \\
-2 \\
9
\end{pmatrix}.
\]

4. Another One—What’s Different?
Consider the system of three linear equations
\[
\begin{align*}
x_1 - 2x_2 + x_3 &= -2 \\
2x_1 - 4x_2 + 8x_3 &= 2 \\
x_1 - 3x_2 - x_3 &= 1.
\end{align*}
\]
a) The linear system is
\[
\begin{align*}
x_1 - 2x_2 + x_3 &= -2 \\
2x_1 - 4x_2 + 8x_3 &= 2 \\
x_1 - 3x_2 - x_3 &= 1.
\end{align*}
\]
By doing two row subtraction operations \((\text{R2} \leftarrow 2\text{R1}) \text{ and } \text{R3} \leftarrow \text{R1}\), we obtain
\[
\begin{align*}
x_1 - 2x_2 + x_3 &= -2 \\
6x_3 &= 6 \\
- x_2 - 2x_3 &= 3.
\end{align*}
\]
b) We swap rows 1 and 2 to obtain
\[
\begin{align*}
x_1 - 2x_2 + x_3 &= -2 \\
- x_2 - 2x_3 &= 3 \\
6x_3 &= 6.
\end{align*}
\]
c) Dividing row 3 by 6 gives \( x_3 = 1 \), which we substitute into the first two equations:
\[
\begin{align*}
x_1 - 2x_2 &= -3 \\
- x_2 &= 5 \\
x_3 &= 1.
\end{align*}
\]
Dividing row 2 by \(-1\) gives \( x_2 = -5 \), which we substitute into the 1st equation:
\[
\begin{align*}
x_1 &= -13 \\
x_2 &= -5 \\
x_3 &= 1.
\end{align*}
\]
This is the solution.
5. Traffic Jam

a) We start with

\[
\begin{align*}
120 + w &= 250 + x \\
120 + x &= 70 + y \\
390 + y &= 250 + z \\
115 + z &= 175 + w
\end{align*}
\]

or

\[
\begin{align*}
x - w &= -130 \\
-x + y &= 50 \\
-y + z &= 140 \\
-z + w &= -60.
\end{align*}
\]

Eliminating \( x \) from the second equation gives

\[
\begin{align*}
x - w &= -130 \\
y - w &= -80 \\
-y + z &= 140 \\
-z + w &= -60.
\end{align*}
\]

b) Eliminating \( y \) from the third equation gives

\[
\begin{align*}
x - w &= -130 \\
y - w &= -80 \\
z - w &= 60 \\
-z + w &= -60.
\end{align*}
\]

c) Eliminating \( z \) from the fourth equation gives

\[
\begin{align*}
x - w &= -130 \\
y - w &= -80 \\
z - w &= 60 \\
0 + 0 &= 0.
\end{align*}
\]

d) We can’t just use substitution, as our final equation is not of the form \( w = (\)\). The number of cars on roads \( x, y, \) and \( z \) all depend on how many cars are on \( w \).

e) The system has infinitely many solutions. There can be as many cars as you want, travelling in a circle around the town.

f) The augmented matrix is

\[
\begin{pmatrix}
1 & 0 & 0 & -1 & -130 \\
0 & 1 & 0 & -1 & -80 \\
0 & 0 & 1 & -1 & 60 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

The pivots are the 1’s. Not every row has a pivot. The fourth column does not have a pivot - as we will discuss in week 3, this means that we can find a solution which makes the fourth variable take any value we want. Such a variable is called a free variable.