1. **Rules of vector SVD**

Which of the following $A = \sigma_1 u_1 v_1^T + \cdots + \sigma_r u_r v_r^T$ are valid singular value decompositions? Why/why not?

   a) $A = 1(1, 0)(1, 0)^T + 3(0, 1)(0, 1)^T$

   b) $A = 4(1, 0)(0, 1)^T + 3(0, 1)(1, 0)^T$

   c) $A = 3(1, -1)(1, 0)^T + 2(1, 1)(0, 1)^T$

   d) $A = -3(1/\sqrt{2}, -1/\sqrt{2}, 0)(1, 0)^T + 2(0, 0, 1)(0, 1)^T$

   e) $A = 3(-1/\sqrt{2}, 1/\sqrt{2}, 0)(1, 0)^T + 2(0, 0, 1)(0, 1)^T$

   f) $A = 5(1, 0, 0)(0, 1)^T + 3(0, 1, 0)(1, 0)^T + 2(0, 0, 1)(0, 1)^T$
2. **The matrix SVD** Suppose that $A$ is an $m \times n$ matrix of rank $r$, with SVD $A = U\Sigma V^T$.

   a) $U$ is a $\, \times \,$ matrix, $\Sigma$ is a $\, \times \,$ matrix, and $V$ is a $\, \times \,$ matrix. The matrices $U$ and $V$ are ______ matrices. The first ___ diagonal entries of $\Sigma$ are $> 0$.

   b) Expand $A^T A$ using $A = U\Sigma V^T$ to see that the matrix $A^T A$ has symmetric diagonalization $Q_1D_1Q_1^T$, with $Q_1 =$ ____ and $D_1 =$ ____. What are the eigenvectors of $A^T A$? What are the eigenvalues?

   c) Expand $AA^T$ using $A = U\Sigma V^T$ to see that the matrix $AA^T$ has symmetric diagonalization $Q_2D_2Q_2^T$, with $Q_2 =$ ____ and $D_2 =$ ____. What are the eigenvectors of $AA^T$? What are the eigenvalues?

   d) Suppose that $i \leq r$. The left singular vector $u_i$ is the $i$th column of $U$, the singular value $\sigma_i$ is the $i$th diagonal entry of $\Sigma$, and the right singular vector $v_i$ is the $i$th column of $V$. Explain why $Av_i = \sigma_i u_i$ by computing $V^T v_i, \Sigma V^T v_i$, and $U\Sigma V^T v_i$. 

3. **Computing the vector SVD**

To find the vector SVD of a matrix $A$:

1. Find the non-zero eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \lambda_r > 0$ of $A^T A$.
2. Find an orthonormal basis of each of the $\lambda_i$ eigenspace of $A^T A$. Listed in order of decreasing eigenvalue, these are the right singular vectors $v_1, \cdots, v_r$.
3. For $i = 1, \ldots, r$, set $\sigma_i = \sqrt{\lambda_i}$ and $u_i = \frac{Av_i}{\sigma_i}$. These are the singular values and left singular vectors.
4. Write $A = \sigma_1 u_1 v_1^T + \cdots + \sigma_r u_r v_r^T$.

Compute the vector SVD of each of the following matrices:

a) $A = \begin{pmatrix} 0 & -1 \\ 3 & 0 \end{pmatrix}$

b) $A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -3 & 0 & 0 \end{pmatrix}$

c) $A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$
4. Computing the matrix SVD

To find the matrix SVD $A = U \Sigma V^T$ of a matrix $A$:

1. Find the symmetric diagonalization $VDV^T$ of $A^TA$, where the eigenvalues are listed in decreasing order: $\lambda_1 \geq \ldots \geq \lambda_n$. The rank $r$ of $A$ is the same as the number of non-zero eigenvalues of $A^TA$ (counted with multiplicity).

2. The columns of $V$ are the right singular vectors $v_1, \ldots, v_r$, followed by an orthonormal basis $v_{r+1}, \ldots, v_n$ of Nul($A)$.

3. For $i = 1, \ldots, r$, set $\sigma_i = \sqrt{\lambda_i}$ and $u_i = \frac{Av_i}{\sigma_i}$. These are the singular values and left singular vectors.

4. We still need the vectors $u_{r+1}, \ldots, u_m$: find these by computing an orthonormal basis of Nul($A^T$) (using RREF to find a basis, Gram–Schmidt to replace it with an orthonormal basis).

5. Finally, the matrix $U$ is the matrix with columns $u_1, \ldots, u_m$, the matrix $V$ was found in (1), and $\Sigma$ has its first $r$ diagonal entries as $\sigma_1, \ldots, \sigma_r$ and the remaining entries of $\Sigma$ being zero.

Compute the matrix SVD of each of the following matrices:

a) $A = \begin{pmatrix} 0 & -1 \\ 3 & 0 \end{pmatrix}$

b) $A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -3 & 0 & 0 \end{pmatrix}$

c) $A = \begin{pmatrix} 1 & 2 \\ 1 & 2 \\ 2 & 4 \end{pmatrix}$
5. **Sums of rank 1 matrices**

This final problem is not about SVDs, but just about sums of rank one matrices.

**a)** Without computing $A$, explain why

$$A = (1, 2, 1)(1, 1)^T + (1, -1, 1)(-1, 1)^T$$

is a rank 2 matrix.

**Hint:** compute $A(1, 1)$ and $A(-1, 1)$, and use this to show that $(1, 2, 1)$ and $(1, -1, 1)$ are in the column space of $A$.

**b)** If $A = u_1v_1^T + \cdots + u_rv_r^T$ for some vectors $u_i \in \mathbb{R}^m$ and $v_j \in \mathbb{R}^n$, explain why the rank of $A$ is at most $r$.

**Hint:** Show that the subspace $\text{Col}(A)$ is contained in the span $\text{Span}\{u_1, \ldots, u_r\}$, which is at most $r$-dimensional.

**c)** If the vectors $u_1, \ldots, u_r \in \mathbb{R}^m$ are a linearly independent set of vectors, and the vectors $v_1, \ldots, v_r \in \mathbb{R}^n$ are also linearly independent, prove that

$$A = u_1v_1^T + \cdots + u_rv_r^T$$

has rank equal to $r$.

**Hint:** Show that there is a vector $v \in \mathbb{R}^n$ which is orthogonal to $v_2, \ldots, v_r$, but $v_1^Tv \neq 0$. Compute $Av$, and use this to show that $u_1 \in \text{Col}(A)$. The same idea shows that $u_2, \ldots, u_r$ are all in $\text{Col}(A)$. 