Homework #8
Answer Key

1. Compute the determinants of the following matrices using Gaussian elimination.

\[
\begin{align*}
\text{a)} \begin{pmatrix} -2 & 1 \\ 1 & 3 \end{pmatrix} & \quad \text{b)} \begin{pmatrix} -3 & 3 & 2 \\ 3 & 0 & 0 \\ -9 & 18 & 7 \end{pmatrix} \\
\begin{pmatrix} -4 & -3 & -3 & -2 \\ 4 & 1 & 2 & -2 \\ -12 & -3 & -9 & 3 \\ 0 & 8 & 19 & 33 \end{pmatrix} & \quad \text{d)} \begin{pmatrix} 2 & 2 & -1 \\ -4 & -5 & 5 \\ 6 & 1 & 12 \end{pmatrix}
\end{align*}
\]

Solution.

a) $-7$ b) $45$ c) $-48$ d) $0$

2. Suppose that

\[
\begin{align*}
\det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = 10 \quad \text{and} \quad \det \begin{pmatrix} a' & b' & c' \\ d & e & f \\ g & h & i \end{pmatrix} = 5.
\end{align*}
\]

Find the determinants of the following matrices.

\[
\begin{align*}
a) \begin{pmatrix} d & e & f \\ a & b & c \\ g & h & i \end{pmatrix} & \quad b) \begin{pmatrix} a & b & c \\ d & e & f \\ g + 2d & h + 2e & i + 2f \end{pmatrix} \\
c) \begin{pmatrix} a & b & c \\ \frac{1}{2}d & \frac{1}{2}e & \frac{1}{2}f \\ g & h & i \end{pmatrix} & \quad d) \begin{pmatrix} a & b & c \\ d & e & f \\ 2g + d & 2h + e & 2i + f \end{pmatrix} \\
e) \begin{pmatrix} a & b & c \\ d & e & f \\ c & f & i \end{pmatrix} & \quad f) \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix} \\
g) \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} & \quad h) \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \\
i) \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}^{-1} & \quad j) \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \\
k) \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}^3 & \quad l) \begin{pmatrix} a & b + 2c & c \\ d & e + 2f & f \\ g & h + 2i & i \end{pmatrix} \\
m) \begin{pmatrix} a + 2a' & b + 2b' & c + 2c' \\ d & e & f \\ g & h & i \end{pmatrix}
\end{align*}
\]

Solution.
3. Find $\det(E)$ when:
   a) $E$ is the elementary matrix for a row replacement.
   b) $E$ is the elementary matrix for $R_i \times c$.
   c) $E$ is the elementary matrix for a row swap.

Solution.

a) 1     b) c    c) $-1$

4. A matrix $A$ has the $PA = LU$ factorization

$$
\begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}
A =
\begin{pmatrix}
2 & 1 & 3 & 0 \\
0 & -1 & 1 & 5 \\
0 & 0 & 4 & 7 \\
0 & 0 & 0 & -3
\end{pmatrix}.
$$

What is $\det(A)$?

Solution.

$\det(A) = 24$

5. Let $A$ be the $n \times n$ matrix ($n \geq 3$) whose $(i, j)$ entry is $i + j$. Use row operations to show that $\det(A) = 0$.

Solution.

$$
\begin{pmatrix}
2 & 3 & 4 & 5 & \cdots \\
3 & 4 & 5 & 6 & \cdots \\
4 & 5 & 6 & 7 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\begin{array}{c}
R_2 \rightarrow R_1 \\
R_3 \rightarrow R_1
\end{array}
\begin{pmatrix}
2 & 3 & 4 & 5 & \cdots \\
1 & 1 & 1 & 1 & \cdots \\
2 & 2 & 2 & 2 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\leftarrow R_3 \rightarrow 2R_2
\begin{pmatrix}
2 & 3 & 4 & 5 & \cdots \\
1 & 1 & 1 & 1 & \cdots \\
0 & 0 & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
$$

This matrix has a row of zeros, and its determinant equals the determinant of $A$.

6. a) Compute the determinants of the matrices in Problem 1 in two more ways: by expanding cofactors along a row, and by expanding cofactors along a column. You should get the same answer using all three methods!
b) Compute the determinants of the matrices in Problem 1(b) and (d) again using Sarrus’ scheme.

c) For the matrix of Problem 1(c), sum the products of the forward diagonals and subtract the products of the backward diagonals, as in Sarrus’ scheme. Did you get the determinant?

Solution.

c) If you try to apply Sarrus’ scheme to this matrix, you get 1 140, which is not equal to its determinant.

7. Consider the matrix

\[ A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{pmatrix}. \]

a) Compute the cofactor matrix \( C \) of \( A \).

b) Compute \( AC^T \). What is the relationship between \( C^T \) and \( A^{-1} \)?

Solution.

a) \( C = \begin{pmatrix} 5 & -2 & 1 \\ -2 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} \)

b) \( AC^T = 3 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad A^{-1} = \frac{1}{\det(A)}C^T \)

8. Consider the \( n \times n \) matrix \( F_n \) with 1’s on the diagonal, 1’s in the entries immediately below the diagonal, and -1’s in the entries immediately above the diagonal:

\[ F_2 = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad F_3 = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \quad F_4 = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \ldots. \]

a) Show that \( \det(F_2) = 2 \) and \( \det(F_3) = 3 \).

b) Expand in cofactors to show that \( \det(F_n) = \det(F_{n-1}) + \det(F_{n-2}) \).

c) Compute \( \det(F_4), \det(F_5), \det(F_6), \det(F_7) \) using b).

This shows that \( \det(F_n) \) is the \( n \)th Fibonacci number. (The sequence usually starts with 1, 1, 2, 3, \ldots, so our \( \det(F_n) \) is the usual \( n + 1 \)st Fibonacci number.)

Solution.

b) First notice that

\[ F_n = \begin{pmatrix} 1 & -1 & 0 & \ldots \\ 1 & 1 & -1 & 0 & \ldots \\ 0 & 1 & 1 & -1 & \ldots \\ \vdots & \vdots & \vdots & \vdots \\ F_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 & \ldots \\ 1 & 1 & -1 & 0 & \ldots \\ 0 & 1 & 1 & -1 & \ldots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 \end{pmatrix}. \]
Expanding cofactors along the first column gives

\[
\det(F_n) = \det(F_{n-1}) - \det \begin{pmatrix}
-1 & 0 & 0 & \cdots \\
1 & 0 & F_{n-2} \\
& \vdots 
\end{pmatrix}.
\]

Expanding cofactors of this last matrix along the first row gives \(\det(F_n) = \det(F_{n-1}) + \det(F_{n-2})\).

c) \(\det(F_4) = 2 + 3 = 5; \ \det(F_5) = 3 + 5 = 8; \ \det(F_6) = 5 + 8 = 13; \ \det(F_7) = 8 + 13 = 21\).

9. Let \(A\) be an \(n \times n\) invertible matrix with integer (whole number) entries.
   a) Explain why \(\det(A)\) is an integer.
   b) If \(\det(A) = \pm 1\), show that \(A^{-1}\) has integer entries.
   c) If \(A^{-1}\) has integer entries, show that \(\det(A) = \pm 1\).

Solution.
   a) If you compute the determinant by expanding cofactors, then you will be adding products of integers together.
   b) By a), the cofactor matrix also has integer entries, so \(A^{-1} = \pm C^T\) has integer entries.
   c) We have \(1 = \det(A)\det(A^{-1})\); if both determinants are integers, then they must be \(\pm 1\).

10. Recall that an orthogonal matrix is a square matrix with orthonormal columns.
   a) Prove that every orthogonal matrix has determinant \(\pm 1\).
   b) Prove that the cofactor matrix of an orthogonal matrix \(Q\) is \(\pm Q\).
   c) Show that \(\begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix}\) is orthogonal, and compute its determinant.
   d) Let \(L\) be the line through \((1, 1, 1)\), and let \(R_L = I_3 - 2P_L\) be the reflection over the plane \(x + y + z = 0\). Compute \(R_L\), show that it is orthogonal, and find its determinant.
   e) Let \(L\) be any line in \(\mathbb{R}^3\), and let \(R_L = I_3 - 2P_L\) be the reflection over the orthogonal plane. Verify that \(R_L\) is orthogonal, and prove that \(\det(R_L) = -1\), as follows: choose an orthonormal basis \(\{u_1, u_2\}\) for \(L^\perp\), and let \(u_3 = u_1 \times u_2\). Show that the matrix \(A\) with columns \(u_1, u_2, u_3\) has determinant 1, and that \(R_LA\) has determinant \(-1\).

[Hint: Recall that \(P_L^2 = P_L = P_L^T\).]

Solution.
a) \(1 = \det(Q^T Q) = \det(Q)^2.\)

b) \(C^T = \det(Q)Q^{-1} = \pm Q^T.\)

c) \(\det\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} = \cos^2 \theta + \sin^2 \theta = 1.\)

d) \(R_L = \frac{1}{3} \begin{pmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{pmatrix}; \ \det(R_L) = -1\)

e) For orthogonality, we have
\[R_L^T R_L = (I_3 - 2P_L^T)(I_3 - 2P_L) = I_3 - 2P_L^T - 2P_L + 4P_L^T P_L = I_3\]
since \(P_L^T = P_L\) and \(P_L^2 = P_L.\)
The determinant of \(A\) is \((u_1 \times u_2) \cdot u_3 = u_3 \cdot u_3 = 1.\) We have
\[R_L u_1 = u_1 \quad R_L u_2 = u_2 \quad R_L u_3 = -u_3\]
since \(u_1, u_2 \in L^\perp\) and \(u_3 \in L.\) Hence \(R_L A\) has columns \(u_1, u_2, -u_3,\) so \(\det(R_L A) = -1.\)

11. Let \(V\) be a subspace of \(\mathbb{R}^n\) and let \(P_V\) be the projection matrix onto \(V.\)

a) Find \(\det(P_V)\) when \(V \neq \mathbb{R}^n.\)

b) Find \(\det(P_V)\) when \(V = \mathbb{R}^n.\)

**Solution.**

a) Recall from Problem 8 on Homework 6 that \(\text{Col}(P_V) = V.\) If \(V \neq \mathbb{R}^n\) then \(P_V\) does not have full row rank, so \(\det(P_V) = 0.\)

b) When \(V = \mathbb{R}^n\) then \(P_V = I_n,\) so \(\det(P_V) = 1.\)

12. Let \(C\) be the hypercube in \(\mathbb{R}^4\) with corners \((\pm 1, \pm 1, \pm 1, \pm 1).\) Compute the volume of \(C.\)

**Solution.**

We translate \(C\) by \((1, 1, 1, 1)\) to become the parallelepiped spanned by \((2, 0, 0, 0),\)
\((0, 2, 0, 0),\) \((0, 0, 2, 0),\) and \((0, 0, 0, 2).\) Its volume is
\[\det\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} = 16.\]

13. Let \(A\) be an \(n \times n\) matrix with columns \(v_1, v_2, \ldots, v_n.\)

a) Show that if \(\{v_1, v_2, \ldots, v_n\}\) is orthogonal then \(|\det(A)| = ||v_1|| ||v_2|| \cdots ||v_n||.\)

[Hint: Compute \(A^T A\) and its determinant.]
b) If \( V \) is a subspace and \( x \) is a vector with orthogonal projection \( x_V \), show that \( \|x_V\| \leq \|x\| \), with equality if and only if \( x \in V \).

c) Show that \( |\det(A)| \leq \|v_1\|\|v_2\| \cdots \|v_n\| \), with equality if and only if the set \( \{v_1, v_2, \ldots, v_n\} \) is orthogonal.

[Hint: Use b) and the QR decomposition of \( A \).]

d) What is the largest possible volume of a parallelepiped spanned by four corners of the hypercube \( C \) of Problem 12?

Solution.

a) Since the columns of \( A \), are orthogonal, we have

\[
A^T A = \begin{pmatrix}
v_1 \cdot v_1 & 0 & \cdots & 0 \\
0 & v_2 \cdot v_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & v_n \cdot v_n
\end{pmatrix}.
\]

Since \( \det(A^T A) = \det(A^T) \det(A) = \det(A)^2 \), this gives

\[
\det(A)^2 = \det(A^T A) = \|v_1\|^2 \|v_2\|^2 \cdots \|v_n\|^2.
\]

b) Let \( x = x_V + x_{V\perp} \) be the orthogonal decomposition of \( x \) with respect to \( V \). Since \( x_V \perp x_{V\perp} \), we have

\[
\|x\|^2 = \|x_V\|^2 + \|x_{V\perp}\|^2 \implies \|x_V\| = \sqrt{\|x\|^2 - \|x_{V\perp}\|^2}.
\]

This shows \( \|x_V\| \leq \|x\| \), and equality holds if and only if \( x_{V\perp} = 0 \).

c) Let \( A = QR \) be the QR decomposition. Since \( Q \) is orthogonal we have \( |\det(Q)| = 1 \) by Problem 10(a), so \( |\det(A)| = |\det(R)| \). Running Gram–Schmidt on the columns \( \{v_1, \ldots, v_n\} \) yields orthogonal vectors \( \{u_1, \ldots, u_n\} \). The matrix \( R \) is upper-triangular with diagonal entries \( \|u_1\|, \ldots, \|u_n\| \), so we have

\[
|\det(A)| = \|u_1\| \|u_2\| \cdots \|u_n\|.
\]

Letting \( V_i = \text{Span}\{u_1, \ldots, u_i\} = \text{Span}\{v_1, \ldots, v_i\} \), we have \( u_{i+1} = (v_{i+1})_{V_i^\perp} \), so by b) we have \( \|u_{i+1}\| \leq \|v_{i+1}\| \), with equality if and only if \( v_{i+1} \perp V_i \).

d) According to c), the largest volume occurs when we choose four orthogonal corners:

\[
\begin{vmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 \\
1 & -1 & 1 & 1 \\
1 & -1 & 1 & -1
\end{vmatrix} = \begin{vmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 \\
1 & -1 & 1 & 1 \\
1 & -1 & 1 & -1
\end{vmatrix} \begin{vmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 \\
1 & -1 & 1 & 1 \\
1 & -1 & 1 & -1
\end{vmatrix} = 16.
\]

14. Compute the area of the triangle pictured below using a \( 2 \times 2 \) determinant. (The grid marks are one unit apart.)
Solution.
Choosing two sides of the triangle, we can express the triangle as half of the parallelogram spanned by those sides:

\[
\begin{bmatrix}
-6 \\ 1
\end{bmatrix}
\begin{bmatrix}
-6 \\ -5 \\ 1 \\ -5
\end{bmatrix}
\]

Hence the area is half the determinant:

\[
\text{area} = \frac{1}{2} \left| \det \begin{pmatrix}
-6 & -5 \\ 1 & -5
\end{pmatrix} \right| = \frac{35}{2}.
\]

15. Consider the parallelepiped \( P \) in \( \mathbb{R}^3 \) spanned by

\[
\begin{align*}
\mathbf{v}_1 &= \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \\
\mathbf{v}_2 &= \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \\
\mathbf{v}_3 &= \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}.
\end{align*}
\]

a) Compute the volume of \( P \) using a triple product \((\mathbf{v}_1 \times \mathbf{v}_2) \cdot \mathbf{v}_3\).

b) Compute the area of each face of \( P \) using cross products.
c) If the “base” of $P$ is the parallelogram spanned by $v_1$ and $v_2$ (blue in the picture), show that the height of $P$ is $\|v_3\| \sin \theta$, where $\theta$ is the angle that $v_3$ makes with the base. (Draw a simpler picture.)

**d)** The volume of $P$ is the area of the base of $P$ times its height. How do you reconcile **c** with **a**? (Remember that $\|u \cdot v\| = \|u\| \|v\| \cos$ (the angle from $u$ to $v$).)

**Solution.**

**a**) The volume of $P$ is

$$\left|(v_1 \times v_2) \cdot v_3\right| = \left|\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}\right| = 4.$$

**b**) The face spanned by $v_i$ and $v_j$ has area $\|v_i \times v_j\|$, as does the parallel face opposite it:

- $\|v_1 \times v_2\| = \sqrt{11}$
- $\|v_1 \times v_3\| = \sqrt{3}$
- $\|v_2 \times v_3\| = \sqrt{11}$.

**c**) This is simple trigonometry:

\[\begin{array}{c}
\left\| v_3 \right\| \sin \theta \\
\left\| v_3 \right\| \\
\text{base}
\end{array}\]

\[\theta\]

\[v_3\]

**d**) The cross product $v_1 \times v_2$ is orthogonal to the base, so this is the correct picture:

\[\begin{array}{c}
\left\| v_3 \right\| \cos \varphi \\
\left\| v_3 \right\| \\
\text{base}
\end{array}\]

Hence the volume is

$$\left\| v_1 \times v_2 \right\| \left\| v_3 \right\| \cos \varphi = \left| (v_1 \times v_2) \cdot v_3 \right|.$$

**16.** Use a cross product to find an implicit equation for the plane

$$V = \text{Span} \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \right\}.$$  

Compare Problem 12(a) in Homework 6.

**Solution.**
We can compute $V^\perp$ using a cross product:

$$V^\perp = \text{Span} \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \times \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \right\} = \text{Span} \left\{ \begin{pmatrix} -3 \\ 6 \\ -3 \end{pmatrix} \right\}.$$ 

Hence $V = \{(x, y, z) \in \mathbb{R}^3 : -3x + 6y - 3z = 0 \}$.

17.  a) Let $v = (a, b)$ and $w = (c, d)$ be vectors in the plane, and let $A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$. By taking the cross product of $(a, b, 0)$ and $(c, d, 0)$, explain how the right-hand rule determines the sign of $\det(A)$.

b) Using the identity

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \times \begin{pmatrix} d \\ e \\ f \end{pmatrix} \cdot \begin{pmatrix} g \\ h \\ i \end{pmatrix} = \det \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix},$$

explain how the right-hand rule determines the sign of a $3 \times 3$ determinant.

Solution.

a) We have

$$\begin{pmatrix} a \\ b \\ 0 \end{pmatrix} \times \begin{pmatrix} c \\ d \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ ad - bc \end{pmatrix}.$$ 

This points up if $\det(A) > 0$ and down otherwise. By the right-hand rule, $\det(A) > 0$ if and only if $u$ is counterclockwise from $v$.

b) The dot product in $(u \times v) \cdot w$ is positive if and only if $w$ makes an acute angle with $u \times v$. This means that, if you apply the right-hand rule to $u, v$, then $w$ points in the general direction of your thumb.

18. Decide if each statement is true or false, and explain why.

a) $\det(A + B) = \det(A) + \det(B)$.

b) $\det(ABC^{-1}) = \frac{\det(A) \det(B)}{\det(C)}$.

c) $\det(AB) = \det(BA)$.

d) $\det(3A) = 3 \det(A)$.

e) If $A^3$ is invertible then $A$ is invertible.

f) The determinant of $A$ is the product of its diagonal entries.

g) If the columns of $A$ are linearly dependent, then $\det(A) = 0$.

h) The determinant of the cofactor matrix of $A$ equals the determinant of $A$.

i) If $A$ is a $3 \times 3$ matrix with determinant zero, then two of the columns of $A$ are scalar multiples of each other.
j) \( u \times v = v \times u \).

k) If \( u \times v = 0 \) then \( u \perp v \).

**Solution.**

a) False.

b) True.

c) True.

d) False: \( \det(3A) = 3^n \det(A) \).

e) True: \( \det(A^5) = \det(A)^5 \).

f) False: this only works for triangular matrices.

g) True.

h) False.

i) False: the columns just have to be coplanar.

j) False: \( u \times v = -v \times u \).

k) False.