Homework #6

due Tuesday, September 29, at 11:59pm

1. Compute a basis for the orthogonal complement of each the following subspaces.
   a) \( \text{Col} \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix} \)
   b) \( \text{Nul} \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix} \)
   c) \( \text{Row} \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix} \)
   d) \( \text{Nul} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \)
   e) \( \text{Span} \left\{ \begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} \right\} \)
   f) \( \text{Col} \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \end{pmatrix} \)

[Hint: solving a)–d) requires only one Gauss-Jordan elimination, and f) doesn’t require any work.]

2. Compute a basis for the orthogonal complement of each the following subspaces.
   a) \( \left\{ (x, y, x) : x, y \in \mathbb{R} \right\} \).
   b) \( \left\{ (x, y, z) \in \mathbb{R}^3 : x = 2y + z \right\} \).
   c) The solution set of the system of equations \( \begin{cases} x + y + z = 0 \\ x - 2y - z = 0. \end{cases} \)
   d) \( \left\{ x \in \mathbb{R}^3 : Ax = 2x \right\} \), where \( A = \begin{pmatrix} 0 & 6 & 8 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \).
   e) The subspace of all vectors in \( \mathbb{R}^3 \) whose coordinates sum to zero.
   f) The intersection of the plane \( x - 2y - z = 0 \) with the \( xy \)-plane.
   g) The line \( \left\{ (t, -t, t) : t \in \mathbb{R} \right\} \).

[Hint: Compare Problem 7 on Homework 5.]

3. For each pair of vectors \( v \) and \( w \), draw \( \text{Span}\{v\} \), and compute and draw the projection \( p \) of \( w \) onto \( \text{Span}\{v\} \).
   a) \( v = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \), \( w = \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} \)
   b) \( v = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \), \( w = \begin{pmatrix} 1 \end{pmatrix} \)

4. For each vector \( v \) in Problem 3, compute the matrix \( P_v \) for projection onto \( V = \text{Span}\{v\} \) using the formula \( P_v = vv^T/v \cdot v \). Verify that \( P_v^2 = P_v \), and that \( P_v w \) is equal to the projection you computed before.

5. For each subspace \( V \) and vector \( b \), compute the orthogonal projection \( b_v \) of \( b \) onto \( V \) by solving a normal equation \( A^T A x = A^T b \), and find the distance from \( b \) to \( V \).
6. For each subspace $V$, compute the orthogonal decomposition $b = b_V + b_{V^\perp}$ of the vector $b = (1, 2, -1)$ with respect to $V$.

   a) $V = \text{Span} \left\{ \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} \right\}$

   b) $V = \text{Nul} \left( \begin{pmatrix} 1 & 2 & 1 \\ -1 & 1 & 0 \\ 2 & 2 & -1 \\ 4 & 3 & 0 \end{pmatrix} \right)$

   c) $V = \mathbb{R}^3$

   d) $V = \{0\}$

   [Hint: Only part a) requires any work.]

7. Compute the orthogonal decomposition $(3, 1, 3) = b_V + b_{V^\perp}$ with respect to each subspace of $V$ of Problem 1(a)–(e).

   [Hint: Only parts a) and c) require any work, and even c) doesn't require work if you're clever enough. In fact, you can solve all five parts by computing two dot products.]

8. Let $P_V$ be the matrix for orthogonal projection onto a subspace $V$ of $\mathbb{R}^n$.

   a) Explain why $P_V^2 = P_V$.

   b) Explain why $P_V + P_{V^\perp} = I_n$.

   c) Explain why $P_V^T = P_V$. (Use the formula $P_V = A(A^TA)^{-1}A^T$.)

   d) What are $\text{Col}(P_V)$ and $\text{Nul}(P_V)$? What is $\text{rank}(P_V)$? Explain your answers.

9. Compute the matrix $P_V$ for orthogonal projection onto each subspace of Problem 5 and Problem 6. Verify properties (a) and (c) of Problem 8.

10. Compute the matrices $P_1, P_2$ for orthogonal projection onto the lines through $a_1 = (-1, 2, 2)$ and $a_2 = (2, 2, -1)$, respectively. Now compute $P_1P_2$, and explain why it is what it is.

11. Consider the plane $V$ defined by the equation $x + 2y - z = 0$. Compute the matrix $P_V$ for orthogonal projection onto $V$ in two ways:
a) Find a basis for $V$, put your basis vectors into a matrix $A$, and use the formula $P_V = A(A^TA)^{-1}A^T$.

b) Compute the matrix for orthogonal projection $P_{V^\perp}$ onto the line $V^\perp$ using the formula $vv^T/v \cdot v$, and subtract: $P_V = I_3 - P_{V^\perp}$.

[Hint: It doesn't take any work to find a basis for $V^\perp$.]

If $V$ is defined by a single equation in 1 000 000 variables, which method do you think a computer would be able to implement?

12. a) Find an implicit equation for the plane

$$\text{Span}\left\{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}\right\}.$$

[Hint: use Problem 1(a).]

b) Find implicit equations for the line $\{(t, -t, t): t \in \mathbb{R}\}$.

[Hint: use Problem 2(g).]

13. Construct a matrix $A$ with each of the following properties, or explain why no such matrix exists.

a) The column space contains $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$, and the null space contains $\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$.

b) The row space contains $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$, and the null space contains $\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$.

c) $Ax = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ is consistent, and $A^T \begin{pmatrix} -1 \\ 2 \end{pmatrix} = 0$.

d) A $2 \times 2$ matrix $A$ with no zero entries such that every row of $A$ is orthogonal to every column.

e) The sum of the columns of $A$ is $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, and the sum of the rows of $A$ is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

14. Suppose that $S$ is a symmetric matrix. Explain why $\text{Col}(S)$ is orthogonal to $\text{Nul}(S)$.

15. Draw the four fundamental subspaces of the following matrices, in grids like below.

a) $\begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}$

b) $\begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}$
16. The floor $V$ and the wall $W$ are not orthogonal subspaces, because they share a nonzero vector (along the line where they meet).

a) If $V = \text{Col}(A)$ and $W = \text{Col}(B)$ for

\[
A = \begin{pmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 2 \end{pmatrix} \quad \quad B = \begin{pmatrix} 5 & 4 \\ 6 & 3 \\ 5 & 1 \end{pmatrix},
\]

find a nonzero vector contained in both $V$ and $W$.

[Hint: You want vectors $x$ and $y$ with $Ax = By$. Form the matrix $(A \ B)$.]

b) Generalize what you did in a) to explain why there do not exist orthogonal planes in $\mathbb{R}^3$.

17. Explain why $A$ has full column rank if and only if $A^T A$ is invertible.

18. Decide if each statement is true or false, and explain why.

a) Two subspaces that meet only at the zero vector are orthogonal.

b) If $A$ is a $3 \times 4$ matrix, then $\text{Col}(A)^\perp$ is a subspace of $\mathbb{R}^4$.

c) If $A$ is any matrix, then $\text{Nul}(A) = \text{Nul}(A^T A)$.

d) If $A$ is any matrix, then $\text{Row}(A) = \text{Row}(A^T A)$.

e) If every vector in a subspace $V$ is orthogonal to every vector in another subspace $W$, then $V = W^\perp$.

f) If $x$ is in $V$ and $V^\perp$, then $x = 0$.

g) If $x$ is in a subspace $V$, then the orthogonal projection of $x$ onto $V$ is $x$. 