Homework #11

due Thursday, November 5, at 11:59pm

1. For each symmetric matrix $S$, find an orthogonal matrix $Q$ and a diagonal matrix $D$ such that $S = QDQ^T$.

   a) \[
   \begin{pmatrix}
   1 & -3 \\
   -3 & 1
   \end{pmatrix}
   \]
   
   b) \[
   \begin{pmatrix}
   1 & -3 \\
   -3 & 9
   \end{pmatrix}
   \]
   
   c) \[
   \begin{pmatrix}
   14 & 2 \\
   2 & 11
   \end{pmatrix}
   \]
   
   d) \[
   \begin{pmatrix}
   7 & 2 & 0 \\
   2 & 6 & 2 \\
   0 & 2 & 5
   \end{pmatrix}
   \]
   
   e) \[
   \begin{pmatrix}
   1 & -8 & 4 \\
   -8 & 1 & 4 \\
   4 & 4 & 7
   \end{pmatrix}
   \]

   The eigenvalues in d) are 3, 6, 9 and in e) are $-9, 9$.

2. For each matrix $S$ of Problem 1, decide if $S$ is positive-semidefinite, and if so, compute its positive-semidefinite square root $\sqrt{S} = Q\sqrt{D}Q^T$. Verify that $(\sqrt{S})^2 = S$.

   **Remark:** Since $\sqrt{S}$ is also symmetric, we have $S = \sqrt{S}^T \sqrt{S}$, so this is another way to factorize a positive-semidefinite matrix as $A^TA$.

3. Consider the matrix

   \[ S = \begin{pmatrix}
   7 & 2 & 0 \\
   2 & 6 & 2 \\
   0 & 2 & 5
   \end{pmatrix} \]

   of Problem 1(d). Write $S$ in the form $\lambda_1q_1q_1^T + \lambda_2q_2q_2^T + \lambda_3q_3q_3^T$ for numbers $\lambda_1, \lambda_2, \lambda_3$ and orthonormal vectors $q_1, q_2, q_3$.

   [Hint: Use the columns of $Q$. Why does this work?]

4. Find **all possible** orthogonal diagonalizations

   \[ \frac{1}{5} \begin{pmatrix}
   41 & 12 \\
   12 & 34
   \end{pmatrix} = QDQ^T. \]

5. Suppose that $A$ is a square matrix such that $A^k = 0$ for some $k > 0$.
   
   a) Show that 0 is the only eigenvalue of $A$.
   
   b) Show that $A = 0$ if it is symmetric.

6. Let $S$ be a symmetric orthogonal $2 \times 2$ matrix.
   
   a) Show that $S = \pm I_2$ if it has only one eigenvalue.
   
   b) Suppose that $S$ has two eigenvalues. Show that $S$ is the matrix for the reflection over a line $L$ in $\mathbb{R}^2$. (Recall that the reflection over a line $L$ is given by $R_L = I_2 - 2P_L$.)

   [Hint: Write $S$ as $\lambda_1q_1q_1^T + \lambda_2q_2q_2^T$, and use the projection formula to write $I_2$ and $P_L$ in this form as well.]
7. a) Let $S$ be a diagonalizable (over $\mathbb{R}$) $n \times n$ matrix with orthogonal eigenspaces: that is, eigenspaces with different eigenvalues are orthogonal subspaces. Prove that $S$ is symmetric.

[Hint: choose orthonormal bases for each eigenspace.]

b) Let $S$ be a matrix that can be written in the form

$$S = \lambda_1 q_1 q_1^T + \lambda_2 q_2 q_2^T + \cdots + \lambda_n q_n q_n^T$$

for some vectors $q_1, q_2, \ldots, q_n$. Prove that $S$ is symmetric.

c) Let $V$ be a subspace of $\mathbb{R}^n$, and let $P_V$ be the projection matrix onto $V$. Use a) or b) to prove that $P_V$ is symmetric. (Compare Problem 8 on Homework 6.)

8. For each symmetric matrix $S$, decide if $S$ is positive-definite. If so, find its $LDL^T$ and Cholesky decompositions. Do not compute any eigenvalues!

a) $\begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}$

b) $\begin{pmatrix} 1 & 2 & 0 \\ 2 & 5 & -1 \\ 0 & -1 & 3 \end{pmatrix}$

c) $\begin{pmatrix} 3 & -2 & 2 \\ -2 & 4 & 0 \\ 2 & 0 & 2 \end{pmatrix}$

d) $\begin{pmatrix} 1 & 1 & 2 & 1 \\ 1 & 3 & 6 & 3 \\ 2 & 6 & 14 & 8 \\ 1 & 3 & 8 & 9 \end{pmatrix}$

e) $\begin{pmatrix} -1 & 2 & 3 & -2 \\ -2 & -3 & -8 & 4 \\ 3 & -8 & -4 & 6 \\ -2 & 4 & 6 & -1 \end{pmatrix}$

9. For which matrices $A$ is $S = A^T A$ positive-definite? If $S$ is not positive-definite, find a vector $x$ such that $x^T S x = 0$. In any case, do not compute $S$!

a) $\begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 0 & 3 \end{pmatrix}$

b) $\begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 3 \end{pmatrix}$

c) $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$

10. a) For each symmetric matrix $S$, compute the associated quadratic form $q(x) = x^T S x$.

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 3 \\ 0 & -1 & 1 \\ 3 & 1 & 0 \end{pmatrix}$$

b) Let $A$ be a square matrix and let $S = \frac{1}{2}(A + A^T)$. Show that $S$ is symmetric and that $x^T A x = x^T S x$. (This is why we only consider symmetric matrices when studying quadratic forms.)

11. For each quadratic form $q(x_1, x_2)$, i) write $q(x)$ in the form $x^T S x$ for a symmetric matrix $S$, ii) find coordinates $y_1, y_2$ such that $q(x) = \lambda_1 y_1^2 + \lambda_2 y_2^2$, iii) draw the solutions of $q(x_1, x_2) = 1$, being sure to draw the shortest and longest solutions, and iv) find the maximum and maximum values of $q(x_1, x_2)$ subject to the constraint
\[ x_1^2 + x_2^2 = 1, \text{ and at which points } (x_1, x_2) \text{ these values are attained.} \]

\[ a) \ q(x_1, x_2) = 14x_1^2 + 4x_1x_2 + 11x_2^2 \quad b) \ q(x_1, x_2) = \frac{1}{10}(21x_1^2 - 6x_1x_2 + 29x_2^2) \]
\[ c) \ q(x_1, x_2) = x_1^2 - 6x_1x_2 + x_2^2 \]

[Hint: An equation of the form \( (x_1/r_1)^2 - (x_2/r_2)^2 = 1 \) defines a hyperbola.]

12. For the quadratic form

\[ q(x_1, x_2, x_3) = 7x_1^2 + 6x_2^2 + 5x_3^2 + 4x_1x_2 + 4x_2x_3, \]
find coordinates \( y_1, y_2, y_3 \) such that \( q(x) = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2 \), and find the maximum and minimum values of \( q(x_1, x_2, x_3) \) subject to the constraint \( x_1^2 + x_2^2 + x_3^2 = 1 \), along with the points \( (x_1, x_2, x_3) \) at which these values are attained.

13. a) If \( S \) is positive-definite and \( C \) is invertible, show that \( CSC^T \) is positive-definite.

b) If \( S \) and \( T \) are positive-definite, show that \( S + T \) is positive-definite.

c) If \( S \) is positive-definite, show that \( S \) is invertible and that \( S^{-1} \) is positive-definite.

[Hint: For a) and b) use the positive-energy characterization of positive-definiteness; for c) use the positive-eigenvalue characterization.]

14. Let \( S \) be a positive-definite matrix.

a) Show that the diagonal entries of \( S \) are positive.

[Hint: compute \( e_i^TSe_i \).]

b) Show that the diagonal entries of \( S \) are all greater than or equal to the smallest eigenvalue of \( S \).

[Hint: if not, apply a) to \( S - aI_n \) for a diagonal entry \( a \) that is smaller than all eigenvalues.]

15. Decide if each statement is true or false, and explain why. All matrices are real.

a) A symmetric matrix is diagonalizable.

b) If \( A \) is any matrix then \( A^TA \) is positive-semidefinite.

c) A symmetric matrix with positive determinant is positive-definite.

d) A positive-definite matrix has the form \( A^TA \) for a matrix \( A \) with full column rank.

e) If \( A = CDC^{-1} \) for a diagonal matrix \( D \) and a non-orthogonal invertible matrix \( C \), then \( A \) is not symmetric.

f) The only positive-definite projection matrix is the identity.

g) All eigenvalues of a positive-definite symmetric matrix are positive real numbers.