Please read all instructions carefully before beginning.

- You have 180 minutes to complete this exam and upload your work. The exam itself is meant to take 75 minutes to complete, so hopefully you will have enough time.

- For full credit you must show your work so that your reasoning is clear.

- If you need clarification or think you’ve found a typo, ask a private question on Piazza. We’ll be monitoring it.

- If you have time, go back and check your work.

- You may use your class notes (not the ones from the website) and the interactive row reducer during this exam. You may use a calculator for doing arithmetic. All other materials and aids are strictly prohibited.

- You are not allowed to receive outside help during this exam. Consulting with someone else is considered cheating; suspected instances will result in immediate referral to the Office of Student Conduct.

- Be sure to tag your answers on Gradescope, and use a scanning app.

- Good luck!

Complete when starting the exam: I will neither give nor receive aid on this exam.

Signed: ___________________________ Time: ______________

Complete after finishing the exam: I have neither given nor received aid on this exam.

Signed: ___________________________ Time: ______________

Miss Wormwood, my dad says when he was in school, they taught him to do math on a slide rule. He says he hasn't used a slide rule since, because he got a five-buck calculator that can do more functions than he could figure out if his life depended on it.

Given the pace of technology, I propose we leave math to the machines and go play outside.

My bills always die in subcommittee.
Problem 1. [20 points]

Consider

\[
A = \begin{pmatrix}
0 & 1 & -1 & 0 \\
-2 & -1 & -1 & 2 \\
2 & 3 & 5 & 4 \\
6 & 3 & -3 & 0
\end{pmatrix}, \quad b = \begin{pmatrix}
-2 \\
-8 \\
4 \\
0
\end{pmatrix}.
\]

a) Carry out Gaussian reduction with maximal partial pivoting to find a \(PA = LU\) decomposition. You should obtain

\[
U = \begin{pmatrix}
6 & 3 & -3 & 0 \\
0 & 2 & 6 & 4 \\
0 & 0 & -4 & -2 \\
0 & 0 & 0 & 3
\end{pmatrix}.
\]

Be sure to specify what \(L\) and \(P\) are. Please write the row operations you performed.

b) Solve the equations \(Ly = Pb\) and \(Ux = y\) to find a solution of \(Ax = b\).

c) Briefly explain why step b) is faster than solving \(Ax = b\) using Gaussian elimination on the augmented matrix \((A | b)\), once you have a \(PA = LU\) decomposition.

Solution.

a) \[
P = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}, \quad L = \begin{pmatrix}
1 & 0 & 0 & 0 \\
\frac{1}{3} & 1 & 0 & 0 \\
0 & \frac{1}{2} & 1 & 0 \\
-\frac{1}{3} & 0 & \frac{1}{2} & 1
\end{pmatrix}
\]

b) \(Ly = Pb \implies y = \begin{pmatrix}
0 \\
4 \\
-4 \\
-6
\end{pmatrix}\) \quad \(Ux = y \implies x = \begin{pmatrix}
1 \\
0 \\
2 \\
-2
\end{pmatrix}\)

c) Gaussian elimination takes about \(\frac{2}{3} \cdot 4^3 \approx 43\) flops, whereas forward- and back-substitution take about \(4^2 = 16\) flops.
Problem 2. [15 points]

a) Compute the inverse of \[
\begin{pmatrix}
1 & -2 & 3 \\
-2 & 6 & -5 \\
2 & 3 & 9
\end{pmatrix}
\]
Be sure to write out any row operations you perform.

b) For which value(s) of \( k \) is
\[
\begin{pmatrix}
1 & -2 & 3 \\
-2 & 6 & k \\
2 & 3 & 9
\end{pmatrix}
\]
not invertible?

c) Suppose that \( A \) is a \( 3 \times 3 \) matrix whose third column is in the span of the first two. Briefly explain why \( A \) is not invertible.
[Hint: can it have full row rank?]

Solution.

\[
\begin{pmatrix}
1 & -2 & 3 \\
-2 & 6 & -5 \\
2 & 3 & 9
\end{pmatrix}^{-1} = \begin{pmatrix}
-69 & -27 & 8 \\
-8 & -3 & 1 \\
18 & 7 & -2
\end{pmatrix}
\]

b) \( k = \frac{-36}{7} \)

c) The column space of \( A \) is a plane in \( \mathbb{R}^2 \) (or a line, or a point), so it does not have full row rank, and hence has fewer than 3 pivots.
Problem 3. [25 points]

Consider
\[
A = \begin{pmatrix}
1 & 3 & -2 & 0 \\
-2 & -6 & 6 & -2 \\
2 & 6 & 3 & -7 \\
\end{pmatrix}, \quad b = \begin{pmatrix}
2 \\
-8 \\
-10 \\
\end{pmatrix}.
\]

a) Find the parametric vector form of the solution set of \(Ax = b\). Be sure to write out any row operations you perform.

b) Write down two different solutions of \(Ax = b\). (Your answer will be two vectors with numbers in them.)

c) Does \(Ax = b'\) have a solution for every vector \(b' \in \mathbb{R}^3\)? Why or why not?

d) Find a spanning set for \(\text{Nul}(A)\).

e) Let \(v = (-1, 1, 1, 1)\). Check that \(v \in \text{Nul}(A)\), and write \(v\) as a linear combination of the spanning vectors you obtained in d).

[Hint: what values do the free variables have to take?]

Solution.

a) \[
\begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
\end{pmatrix} = x_2 \begin{pmatrix}
-3 \\
1 \\
0 \\
0 \\
\end{pmatrix} + x_4 \begin{pmatrix}
2 \\
0 \\
1 \\
1 \\
\end{pmatrix} + \begin{pmatrix}
-2 \\
0 \\
0 \\
0 \\
\end{pmatrix}
\]

b) Choose any values of the free variables. For instance, \((x_2, x_4) = (1, 0)\) and \((0, 1)\) give \((-5, 1, -2, 0)\) and \((0, 0, -1, 1)\), respectively.

c) No: the matrix \(A\) has only two pivots, hence does not have full row rank.

d) \[
\text{Nul}(A) = \text{Span} \left\{ \begin{pmatrix}
-3 \\
1 \\
0 \\
0 \\
\end{pmatrix}, \begin{pmatrix}
2 \\
0 \\
1 \\
1 \\
\end{pmatrix} \right\}
\]

e) One computes \(Av = 0\), so \(v \in \text{Nul}(A)\). The second (resp. fourth) coordinate of \(v\) is the value of \(x_2\) (resp. \(x_4\)), so

\[
v = \begin{pmatrix}
-3 \\
1 \\
0 \\
0 \\
\end{pmatrix} + \begin{pmatrix}
2 \\
0 \\
1 \\
1 \\
\end{pmatrix}.
\]
**Problem 4.**

For a certain $2 \times 2$ matrix $A$, the solution set of $Ax = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is drawn. Copy this grid onto your paper, and draw a) the solution set of $Ax = 0$ and b) the solution set of $Ax = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$. Be sure to label which is which.

\[
Ax = \begin{pmatrix} 1 \\ 1 \end{pmatrix}
\]

- c) What is the rank of $A$?
- d) Draw the column space of $A$ in a grid like below. Be precise!

**Solution.**

a) and b)

\[
Ax = \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \quad Ax = 0, \quad Ax = \begin{pmatrix} 1 \\ 1 \end{pmatrix}
\]

c) The rank is 1, since $A$ has one free variable.

d)
Problem 5. [15 points]

Find examples of matrices with the following properties. If no such matrix exists, write “no way, man,” or use your favorite colloquialism instead.

a) A matrix A, in RREF, such that Ax = b has at least one solution for every b, but A does not have full column rank.

b) A 3 × 5 matrix of rank 4, in RREF.

c) A 2 × 2 matrix A such that the solution set of Ax = \begin{pmatrix} 3 \\ 4 \end{pmatrix} is a line, and Ax = \begin{pmatrix} 1 \\ 1 \end{pmatrix} has no solutions.

d) A 3 × 3 matrix A with no zero entries, such that Col(A) is a plane.

e) A 4 × 4 matrix A with full row rank such that A(1, 2, −1, 1) = 0.

Solution.

a) There are many answers; one is \( A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \).

b) Yeah, right.

c) There are many answers; one is \( A = \begin{pmatrix} 3 & 0 \\ 4 & 0 \end{pmatrix} \).

d) There are many answers; one is \( A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix} \).

e) As if.
Problem 6. [18 points]

Which of the following are subspaces of \( \mathbb{R}^4 \)? If not, why?

a) \( \text{Span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 2 \\ 0 \\ -2 \\ -1 \\ -1 \\ 2 \end{pmatrix} \right\} \)

b) \( \text{Nul} \left( \begin{array}{cccc} 2 & -1 & 3 & \\
0 & 0 & 4 & \\
6 & -4 & 2 & \\
-9 & 3 & 4 & \\
\end{array} \right) \)

c) \( \text{Col} \left( \begin{array}{cccc} 2 & -1 & 3 & \\
0 & 0 & 4 & \\
6 & -4 & 2 & \\
-9 & 3 & 4 & \\
\end{array} \right) \)

d) \( \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\} \)

e) \( \{\} \)

f) \( V = \left\{ \text{all vectors } \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \text{ in } \mathbb{R}^4 \text{ such that } xy = zw \right\} \)

Solution.

a) Yes.

b) No: this is a subspace of \( \mathbb{R}^3 \).

c) Yes.

d) Yes.

e) No: this does not contain the zero vector.

f) No: this is not closed under addition. For instance,

\[
\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \end{pmatrix}.
\]