Please **read all instructions** carefully before beginning.

- You have 180 minutes to complete this exam and upload your work. The exam itself is meant to take 75 minutes to complete, so hopefully you will have enough time.
- If you have time, go back and check your work.
- You may use **your class notes** (not the ones from the website) and the **interactive row reducer** during this exam. All other materials are strictly prohibited.
- You are not allowed to receive **outside help** during this exam. Consulting with someone else is considered cheating; suspected instances will result in immediate referral to the Office of Student Conduct.
- For full credit you must show your work so that your reasoning is clear.
- Be sure to **tag your answers** on Gradescope, and **use a scanning app**.
- Good luck!

**Complete when starting the exam:** I will neither give nor receive aid on this exam.

Signed: ___________________________ Time: ________________

**Complete after finishing the exam:** I have neither given nor received aid on this exam.

Signed: ___________________________ Time: ________________

This is a practice exam. It is meant to be similar in format, length, and difficulty to the real exam. It is **not** meant as a comprehensive list of study problems. I recommend completing the practice exam in 75 minutes, without distractions.
Problem 1. [25 points]

Consider

\[ A = \begin{pmatrix}
  1 & 2 & 5 & 0 \\
  1 & 2 & 4 & 2 \\
  0 & -1 & 0 & 8 \\
 -1 & -3 & -1 & -1 \\
\end{pmatrix}, \quad b = \begin{pmatrix}
  12 \\
  1 \\
 -30 \\
  6 \\
\end{pmatrix}. \]

The goal of this question is to solve the equation \( Ax = b \) like a computer would.

a) Carry out Gaussian reduction with maximal partial pivoting to find a \( PA = LU \) decomposition. You should obtain

\[ U = \begin{pmatrix}
  1 & 2 & 5 & 0 \\
  0 & -1 & 0 & 8 \\
  0 & 0 & 4 & -9 \\
  0 & 0 & 0 & -\frac{1}{4} \\
\end{pmatrix}. \]

Be sure to specify what \( L \) and \( P \) are.

b) Solve the equations \( Ly = Pb \) and \( Ux = y \) to find a solution of \( Ax = b \).

Problem 2. [5 points]

Consider the matrix

\[ B = \begin{pmatrix}
  1 & -1 & 0 \\
 -1 & 7 & 1 \\
 2 & 4 & 1 \\
\end{pmatrix}. \]

Is \( B \) invertible? If so, find its inverse. If not, explain why.
Problem 3. [20 points]

Consider the matrix

\[ D = \begin{pmatrix} 1 & 2 & 3 & 2 & 14 & 9 \\ 0 & 0 & 0 & 2 & 10 & 6 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix}. \]

Note that this matrix is in row echelon form.

a) Fill in the blanks:
   (1) The column space Col(D) is a subspace of \( \mathbb{R}^m \), where \( m = \) ___.
   (2) The null space Nul(D) is a subspace of \( \mathbb{R}^n \), where \( n = \) ___.

b) Write down a vector \( b \) such that \( Dx = b \) has no solution. If no such vector exists, explain why not.

c) Compute the reduced row echelon form of \( D \).

d) Find a set of vectors that spans Nul(D).

e) Compute the solutions of \( Dx = (2, 2, 0) \), noting that \( D(0, 0, 0, 1, 0, 0) = (2, 2, 0) \).

Problem 4. [5 points]

Decide if a matrix with the following properties has full row rank, full column rank, both, or neither.

a) \( Ax = b \) has 0 or 1 solutions, depending on \( b \).

b) \( Ax = b \) has 1 solution for every \( b \).

c) \( Ax = b \) has 0 or \( \infty \) solutions, depending on \( b \).

d) \( Ax = b \) has \( \infty \) solutions, for every \( b \).
Problem 5.\[10\text{ points}\]

Consider the subspace $V$ and vectors $b_1$ and $b_2$:

$$V = \text{Span}\left\{ \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ -5 \\ -4 \end{pmatrix} \right\}, \quad b_1 = \begin{pmatrix} -2 \\ 8 \\ 1 \end{pmatrix}, \quad b_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

a) Is $b_1$ contained in $V$? If so, write $b_1$ as a linear combination of the vectors in the span; if not, explain why.

b) Same question for $b_2$.

c) Circle one: $V$ is a point line plane space

Why?

Problem 6.\[10\text{ points}\]

Give examples of $2 \times 2$ matrices $A, B, C$ with ranks 0, 1, and 2, respectively, and draw pictures of the null space and column space. (Be precise!)

a) Rank 0: $A = \begin{pmatrix} \end{pmatrix}$

b) Rank 1: $B = \begin{pmatrix} \end{pmatrix}$

c) Rank 2: $C = \begin{pmatrix} \end{pmatrix}$