Please **read all instructions** carefully before beginning.

- You have 200 minutes to complete this exam and upload your work. The exam itself is meant to take 100 minutes to complete, so hopefully you will have enough time.

- For full credit you must **show your work** so that your reasoning is clear.

- If you need clarification or think you’ve found a typo, ask a **private question on Piazza**. We’ll be monitoring it.

- If you have time, go back and check your work.

- You may use **your class notes** (not the ones from the website) and the **interactive row reducer** during this exam. You may use a **calculator** for doing arithmetic. All other materials and aids are strictly prohibited.

- You are not allowed to receive **outside help** during this exam. Consulting with someone else is considered cheating; suspected instances will result in immediate referral to the Office of Student Conduct.

- Be sure to **tag your answers** on Gradescope, and **use a scanning app**.

- Good luck!

**Complete when starting the exam:** I will neither give nor receive aid on this exam.

Signed: ___________________________  Time: ______________

**Complete after finishing the exam:** I have neither given nor received aid on this exam.

Signed: ___________________________  Time: ______________

This is a practice exam. It is meant to be similar in format, length, and difficulty to the real exam. It is **not** meant as a comprehensive list of study problems. I recommend completing the practice exam in 75 minutes, without distractions.
Problem 1.  

Consider the sequence of numbers 0, 1, 5, 31, 185, . . . given by the recursive formula

\[ a_0 = 0 \quad a_1 = 1 \quad a_n = 5a_{n-1} + 6a_{n-2}. \]

a) Find a matrix \( A \) such that

\[
\begin{pmatrix}
    a_{n-2} \\
    a_{n-1}
\end{pmatrix}
= \begin{pmatrix}
    a_{n-1} \\
    a_n
\end{pmatrix}.
\]

b) Find the eigenvalues of \( A \), and find corresponding eigenvectors.

c) Give a non-recursive formula for \( a_n \).

Solution.

a) The matrix is \( A = \begin{pmatrix} 0 & 1 \\ 6 & 5 \end{pmatrix} \).

b) The eigenvalues are \( \lambda_1 = 6 \) and \( \lambda_2 = -1 \), with corresponding eigenvectors \( w_1 = \begin{pmatrix} 1 \\ 6 \end{pmatrix} \) and \( w_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \).

c) First we write \( \begin{pmatrix} a_n \\ a_{n+1} \end{pmatrix} \) in terms of our eigenbasis:

\[
\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{7}(w_1 + w_2).
\]

Hence we have

\[
\begin{pmatrix} a_n \\ a_{n+1} \end{pmatrix} = A^n \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \frac{1}{7} \left( 6^n w_1 + (-1)^n w_2 \right) = \frac{6^n}{7} \begin{pmatrix} 1 \\ 6 \end{pmatrix} + \frac{(-1)^n}{7} \begin{pmatrix} -1 \\ 1 \end{pmatrix}.
\]

The first coordinate is

\[
a_n = \frac{1}{7} \left( 6^n - (-1)^n \right).
\]
Problem 2. [20 points]

A certain matrix $A$ has singular value decomposition $A = U\Sigma V^T$, where

$$U = \begin{pmatrix} u_1 & u_2 & u_3 & u_4 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} v_1 & v_2 & v_3 & v_4 & v_5 \end{pmatrix}. $$

a) What is the rank of $A$?

b) What is the maximum value of $\|Ax\|$ subject to $\|x\| = 1$?

c) Find orthonormal bases of the four fundamental subspaces of $A$.

d) What is the singular value decomposition of $A^T$?

e) What is the pseudoinverse of $A$?

Solution.

a) $A$ has rank 3.

b) $\|Av_1\| = 4$.

c) $\text{Nul}(A) \setminus \{v_4, v_5\}$, $\text{Col}(A) \setminus \{u_1, u_2, u_3\}$, $\text{Nul}(A^T) \setminus \{u_4\}$, $\text{Row}(A) \setminus \{v_1, v_2, v_3\}$

d) $A^T = V\Sigma^T U^T$

e) $A^+ = V \begin{pmatrix} 1/4 & 0 & 0 & 0 \\ 0 & 1/3 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} U^T$
Problem 3. \[15 \text{ points}\]

Consider the matrix
\[
A = \begin{pmatrix}
1 & 2 & 5 & 0 \\
1 & 2 & 4 & 2 \\
0 & -1 & 0 & 8 \\
-1 & -3 & -1 & -1
\end{pmatrix}.
\]

a) Find a permutation matrix $P$, a lower-unitriangular matrix $L$, and an upper-triangular matrix $U$ such that $PA = LU$.

b) Use a) to solve $Ax = b$, for $b = \begin{pmatrix} 12 \\ 1 \\ -30 \\ 6 \end{pmatrix}$.

c) What is $\det(A)$?

Solution.

a) \[
P = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad L = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 \\
-1 & 1 & -4 & 1
\end{pmatrix}, \quad U = \begin{pmatrix}
1 & 2 & 5 & 0 \\
0 & -1 & 0 & 8 \\
0 & 0 & -1 & 2 \\
0 & 0 & 0 & -1
\end{pmatrix}
\]

b) \[
x = \begin{pmatrix}
1 \\
-2 \\
3 \\
-4
\end{pmatrix}
\]

c) $\det(A) = 1$
Problem 4. [20 points]

Consider the subspace

\[ W = \text{Span}\left\{ \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix} \right\}. \]

a) Compute an orthonormal basis for \( W \). [Hint: \( W \) is not all of \( \mathbb{R}^3 \).]

b) What is \( \dim(W) \)?

c) Compute the matrix \( P \) for orthogonal projection onto \( W \). (You may write \( P \) as a product of two matrices, without expanding.)

d) Write an eigenvector of \( P \).

e) Find the distance from \( \begin{pmatrix} 1 \\ 5 \\ -2 \end{pmatrix} \) to \( W \).

f) Compute a basis for \( W^\perp \).

Solution.

a) \[ \left\{ \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}, \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \right\} \]

b) \( \dim(W) = 2 \)

c) \[ P = \begin{pmatrix} 1/\sqrt{6} & 2/\sqrt{5} \\ -1/\sqrt{6} & 0 \\ -2/\sqrt{6} & 1/\sqrt{5} \end{pmatrix} \begin{pmatrix} 1/\sqrt{6} & -1/\sqrt{6} & -2/\sqrt{6} \\ 2/\sqrt{5} & 0 & 1/\sqrt{5} \end{pmatrix} \]

d) Any nonzero vector in \( W \); for instance, \( \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \).

e) Noting that \[ \begin{pmatrix} 1 \\ 5 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} = 0 \]

shows \(1, 5, -2)\) is in Nul(\(A\)), so its projection is zero, and hence the distance is just \[ \left\| \begin{pmatrix} 1 \\ 5 \\ -2 \end{pmatrix} \right\| = \sqrt{30}. \]

f) We showed above that the vector \(1, 5, -2)\) is in \(W^\perp\). Since \( \dim(W) = 2 \) we have \( \dim(W^\perp) = 1 \), so it is a basis.
Problem 5. [20 points]

Consider the data points
\[
\begin{pmatrix}
3 \\
2 \\
-2 \\
1 \\
2 \\
2 \\
-2 \\
4 \\
-4 \\
2 \\
0 \\
-4
\end{pmatrix}.
\]

a) Form the matrix $A_0$ with the data points as columns, and form the matrix $A$ by subtracting the row averages from $A_0$.

b) Find the eigenvalues and eigenvectors of $S = \frac{1}{3}AA^T$.

c) Find the line closest to the columns of $A$.

d) Find the plane closest to the columns of $A$.

e) Find the plane closest to the original data points.

Solution.

a) $A_0 = \begin{pmatrix}
3 & 1 & 2 & 2 \\
2 & 2 & 4 & 0 \\
-2 & -2 & -4 & -4
\end{pmatrix}$, $A = \begin{pmatrix}
1 & -1 & 0 & 0 \\
0 & 0 & 2 & -2 \\
1 & 1 & -1 & -1
\end{pmatrix}$

b) The matrix
\[
S = \frac{1}{3}AA^T = \frac{1}{3} \begin{pmatrix}
2 & 0 & 0 \\
0 & 8 & 0 \\
0 & 0 & 4
\end{pmatrix}
\]

is diagonal. It has eigenvalues $8/3, 4/3, 2/3$ with eigenvectors $e_2, e_3, e_1$, respectively.

c) The closest line is spanned by $e_2$: it is the $y$-axis.

d) The closest plane is spanned by $e_2, e_3$: it is the $yz$-plane.

e) We need to add back the row averages:
\[
\begin{pmatrix}
2 \\
2 \\
-3
\end{pmatrix} + \text{Span}\{e_2, e_3\}.
\]
Problem 6. [30 points]

Give examples of matrices with each of the following properties. If no such matrix exists, explain why. All matrices in this problem have real entries.

a) A $4 \times 4$ matrix $A$ such that $\text{Col}(A) = \text{Nul}(A)$.

b) A $4 \times 6$ matrix of rank 6.

c) A $2 \times 2$ matrix whose column space is the line $3x + y = 0$ and with null space $\{0\}$.

d) A $2 \times 2$ matrix $A$ that is not diagonalizable over $\mathbb{C}$, such that $A^2$ is diagonalizable.

e) A $3 \times 4$ matrix with singular values 2 and 1.

f) A positive-semidefinite symmetric matrix that is not positive-definite.

g) A matrix of rank 1 that cannot be written as a product of a column vector and a row vector.

h) A nonzero symmetric matrix with characteristic polynomial $p(\lambda) = \lambda^2$.

i) A matrix $A$ satisfying $\text{dim}(\text{Row}(A)^\perp) = 2$ and $\text{dim}(\text{Col}(A)^\perp) = 3$.

j) A $3 \times 3$ matrix with no real eigenvalues.

Solution.

a) One example is 
\[
\begin{pmatrix}
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 1 \\
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

b) Does not exist: the rank is at most 4.

c) Does not exist: $\text{dim Col}(A) + \text{dim Nul}(A) = 2$.

d) One example is \[
\begin{pmatrix}
  0 & 1 \\
  0 & 0 \\
\end{pmatrix}
\]

e) One example is 
\[
\begin{pmatrix}
  2 & 0 & 0 & 0 \\
  0 & 1 & 0 & 0 \\
  0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

f) One example is \[
\begin{pmatrix}
  1 & 0 & 0 \\
  0 & 0 & 0 \\
\end{pmatrix}
\]

g) Does not exist by the outer product form of the SVD.

h) Does not exist: any such matrix equals $QDQ^T$ for $D = 0$.

i) One example is 
\[
\begin{pmatrix}
  1 & 0 & 0 \\
  0 & 0 & 0 \\
  0 & 0 & 0 \\
\end{pmatrix}
\]

j) Does not exist: every cubic polynomial has a real root.
Problem 7. [10 points]

Let $A$ be an $m \times n$ matrix. Which of the following are equivalent to the statement “the columns of $A$ are linearly independent?” Circle all that apply.

1. $A$ has full column rank.
2. $Ax = b$ has a unique solution for every $b$ in $\mathbb{R}^m$.
3. $Ax = b$ has a unique least-squares solution for every $b$ in $\mathbb{R}^m$.
4. $Ax = 0$ has a unique solution.
5. $A$ has $n$ pivots.
6. $\text{Nul}(A) = \{0\}$.
7. $m \geq n$.
8. $A^T A$ is invertible.
9. $AA^T$ is invertible.
10. $A^+ A$ is the identity matrix.
11. $\text{Row}(A) = \mathbb{R}^n$.

Solution.

(1), (3), (4), (5), (6), (8), (10), (11)
Problem 8. [10 points]

A certain $2 \times 2$ matrix $A$ has the singular value decomposition

$$A = \begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix}^T,$$

where $u_1, u_2, v_1, v_2$ are drawn in the diagrams below. Given $x$ and $y$ in the diagram on the left, draw $Ax$ and $Ay$ on the diagram on the right.

![Diagram](image.png)

Problem 9. [15 points]

A certain diagonalizable $2 \times 2$ matrix $A$ is equal to $CDC^{-1}$, where $C$ has columns $w_1, w_2$ pictured below, and $D = \begin{pmatrix} 2 & 0 \\ 0 & 1/4 \end{pmatrix}$.

a) Draw $C^{-1}v$ on the left.

b) Draw $DC^{-1}v$ on the left.

c) Draw $Av = CDC^{-1}v$ on the right.

![Diagram](image.png)