Please **read all instructions** carefully before beginning.

- Do not open this test booklet until you are directed to do so.
- You have 75 minutes to complete this exam.
- If you finish early, go back and check your work.
- This exam is closed book.
- You may use a calculator to do arithmetic, but you should not need one. No other technology is allowed.
- For full credit you must show your work so that your reasoning is clear.
- Do not spend too much time on any one problem. Read them all through first and attack them in an order that allows you to make the most progress.
- Good luck!

This is a practice exam. It is meant to be similar in format, length, and difficulty to the real exam. It is **not** meant as a comprehensive list of study problems. I recommend completing the practice exam in 75 minutes, without notes or distractions.
Problem 1. [25 points]

Consider

\[
A = \begin{pmatrix}
3 & -3 \\
3 & 1 \\
3 & 5 \\
3 & 1
\end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix}
2 \\
4 \\
6 \\
8
\end{pmatrix}.
\]

a) Use Gram–Schmidt to find the QR decomposition of \( A \). You should get

\[
Q = \begin{pmatrix}
1/2 & -1/\sqrt{2} \\
1/2 & 0 \\
1/2 & 1/\sqrt{2} \\
1/2 & 0
\end{pmatrix}.
\]

b) Find the least squares solution \( \hat{x} \) of \( Ax = v \), using your QR decomposition above or otherwise.

c) Find the projection \( p \) of \( v \) onto \( C(A) \).
Problem 2.

Consider

\[ A = \begin{pmatrix} 1 & 3 & 1 & 2 \\ 2 & 6 & 4 & 8 \\ 0 & 0 & 2 & 4 \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}. \]

a) Find the general solution of the system of equations \( Ax = b \).

b) \( \dim(N(A)) = \) ___
Problem 3.

a) Suppose that $V$ and $W$ are subspaces of $\mathbb{R}^n$ with $\dim(V) + \dim(W) > n$. Show that there is a nonzero vector contained in both $V$ and $W$.

b) If $P$ is the projection matrix onto the row space of a matrix $A$, explain why $I - P$ projects vectors onto the null space of $A$.

c) Explain why any set containing the zero vector is linearly dependent.
Problem 4.

Let $A$ be a $4 \times 5$ matrix whose null space is spanned by

$$a = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 3 \\ 1 \end{pmatrix}.$$

a) Find the projection matrix onto the null space of $A$.

b) Find the projection matrix onto the row space of $A$. 
Problem 5.

Consider the subspace

\[ V = \{ \text{all solutions of } 2x + 3y + 4z = 0 \}. \]

a) Find a basis for \( V \).

b) Find a basis for \( V^\perp \).
Problem 6.

A subspace $V$ and a vector $v$ are drawn below. Draw the projection $p$ of $v$ onto $V$, and draw the projection $p_\perp$ of $v$ onto $V^\perp$. Label your answers!