1. \( A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 3 & 1 \\ 4 & 1 \end{pmatrix} \), \( \vec{b} = \begin{pmatrix} 0 \\ 8 \\ 8 \\ 20 \end{pmatrix} \). So \( A^T A \hat{x} = \begin{pmatrix} 26 & 8 \\ 8 & 4 \end{pmatrix} \) and \( A^T \vec{b} = \begin{pmatrix} 112 \\ 36 \end{pmatrix} \). \( \hat{x} = (A^T A)^{-1} A^T \vec{b} = \begin{pmatrix} 1 \\ 5 \\ 13 \\ 17 \end{pmatrix} \). So \( \vec{p} = A \hat{x} = \begin{pmatrix} 1 \\ 8 \\ 8 \\ 20 \end{pmatrix} \), and \( E = |\vec{e}|^2 = (\vec{b} - \vec{p})^2 = 44 \).

5. \( A = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \). So \( A^T A = 4 \), and \( A^T \vec{b} = 36 \). Therefore \( \hat{x} = 9 \). Note that this is the average of the \( y \) values.

9. \( A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{pmatrix} \). \( \hat{x} = (A^T A)^{-1} A \vec{b} = \begin{pmatrix} 2 \\ \frac{4}{3} \\ \frac{16}{7} \end{pmatrix} \). Best fit parabola is \( 2 + \frac{4}{3} x + \frac{2}{3} x^2 \), and \( \vec{p} = \begin{pmatrix} 2 \\ 4 \\ \frac{12}{18} \end{pmatrix} \).

12. (a) \( \vec{a}^T \vec{a} \hat{x} = \vec{a}^T \vec{b} \) becomes \( n \hat{x} = \sum_{i=1}^{m} b_i \), so \( \hat{x} = \frac{1}{n} \sum_{i=1}^{m} b_i \) (the average of the \( b_i \)'s).
(b) \( \vec{e} = \vec{b} - \vec{a} \hat{x} = \begin{pmatrix} b_1 - \hat{x} \\ \vdots \\ b_m - \hat{x} \end{pmatrix} \). So the variance \( |\vec{e}|^2 = \sum_{i=1}^{m} (b_i - \hat{x})^2 \), and the standard deviation is \( \sqrt{\sum_{i=1}^{m} (b_i - \hat{x})^2} \).

(c) \( \vec{e} = \vec{b} - \vec{p} = \begin{pmatrix} -2 \\ -1 \\ 3 \end{pmatrix} \). \( \vec{e} \cdot \vec{p} = -2 \cdot 3 + -1 \cdot 3 + 3 \cdot 3 = 0 \), so \( \vec{e} \perp \vec{p} \). \( P = \frac{\vec{a} \vec{a}^T}{\vec{a} \vec{a}^T} = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \).

17. \( 7 = C - D, 7 = C + D, 21 = C + 2D \). Then \( A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \) and \( \vec{b} = \begin{pmatrix} 7 \\ 7 \\ 21 \end{pmatrix} \). So \( \hat{x} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} A^{T} A^{-1} A^{T} \vec{b} = \begin{pmatrix} 9 \\ 4 \end{pmatrix} \). The best fit line is \( y = 4x + 9 \).

18. \( \vec{p} = A \hat{x} = \begin{pmatrix} 5 \\ 13 \\ 17 \end{pmatrix} \). \( \vec{e} = \vec{p} - \vec{b} = \begin{pmatrix} 2 \\ 6 \\ 4 \end{pmatrix} \). \( P \vec{e} = 0 \), since by definition \( \vec{e} \in N(P) \).

19. We are solving \( A^{T} \hat{x} = A^{T} \vec{c} \) for \( A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \), \( \vec{c} = [2, -6, 4]^T \). RREF of \( [A^{T} A | A^{T} \vec{c}] \) is 
\( \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \), so \( \hat{x} = [0, 0]^T \). The line is therefore \( y = 0 \). This makes sense, since \( \vec{c} \perp [-1, 1, 2]^T \), the vector of times, so the projection onto \( C(A) \) is the zero vector.

25. Consider the matrix \( \begin{pmatrix} 1 & t_1 \\ 1 & t_2 \\ 1 & t_3 \end{pmatrix} b_1 \). The row echelon form of this is
\[
\begin{pmatrix}
1 & t_1 & | & b_1 \\
0 & 1 & | & \frac{b_2-b_1}{t_2-t_1} \\
0 & 0 & | & \frac{b_3-b_1}{(t_3-t_1)} (t_3 - t_1)
\end{pmatrix}
\]
So we need \((b_3 - b_1) - \frac{t_3-t_1}{t_2-t_1} (b_2 - b_1) = 0\), or
\[
\frac{b_3 - b_1}{t_3 - t_1} = \frac{b_2 - b_1}{t_2 - t_1}.
\]
That is, the slope between \((t_1, b_1)\) and \((t_2, b_2)\) must be the same as the slope between \((t_1, b_1)\) and \((t_3, b_3)\)....which makes sense!

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4. (a) \[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix}
\]. Any non-square matrix with orthonormal columns will do.

(b) Any vector and the the zero vector.

(c) \[
\frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}
\]. There are many other answers.

6. \((Q_1Q_2)^T(Q_1Q_2) = Q_2^TQ_1^TQ_1Q_2 = Q_2^TQ_2 = I\).

10. (a) If \(c_1\bar{q}_1 + c_2\bar{q}_2 + c_3\bar{q}_3 = \bar{0}\), and the \(q_i\)'s are orthonormal, then by taking dot product with \(q_1\), we get \(c_1\bar{q}_1 \cdot \bar{q}_1 + c_2\bar{q}_2 \cdot \bar{q}_1 + c_3\bar{q}_3 \cdot \bar{q}_1 = \bar{0} \cdot \bar{q}_1 \Rightarrow c_1 = 0\). Similarly, by taking dot products with \(q_2\) and \(q_3\), we get that \(c_2 = c_3 = 0\).

(b) If \(Q\bar{x} = \bar{0}\), then \(Q^TQ\bar{x} = Q^T\bar{0} = \bar{0}\). Since \(Q\) is orthonormal, \(Q^TQ = I\), so we get \(\bar{x} = \bar{0}\).

11. (a) First normalize \(\vec{a}\) to get \(q_1 = \frac{1}{10} \begin{pmatrix} 1 \\ 3 \\ 4 \\ 5 \\ 7 \end{pmatrix}\). Then \(v_2 = \bar{b} - \bar{q}_1^T\bar{b}\bar{q}_1 = \begin{pmatrix} -7 \\ 3 \\ 4 \\ -5 \\ 1 \end{pmatrix}\). Normalize to get \(q_2 = \frac{1}{10} \begin{pmatrix} -7 \\ 3 \\ 4 \\ -5 \\ 1 \end{pmatrix}\).

(b) We want to project \(\vec{v} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}\) onto the plane. Let \(Q = [q_1|q_2]\). Then \(P = QQ^T = \)
\[
\frac{1}{50} \begin{pmatrix} 25 & -9 & -12 & 20 & 0 \\
-9 & 9 & 12 & 0 & 12 \\
-12 & 12 & 16 & 0 & 16 \\
20 & 0 & 0 & 25 & 15 \\
0 & 12 & 16 & 15 & 25 \end{pmatrix}
\]. So \(P\vec{v} = \frac{1}{50} \begin{pmatrix} 25 \\ -9 \\ -12 \\ 20 \\ 0 \end{pmatrix}\).
15. (a) By GS on the matrix $[A|e_1]$ where $e_1 = [1, 0, 0]^T$, we get $q_1 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$, $q_2 = \frac{1}{\sqrt{66}} \begin{pmatrix} 7 \\ 1 \\ 4 \end{pmatrix}$, $q_3 = \frac{1}{\sqrt{11}} \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}$.

(b) $q_3 \in N(A^T)$.

(c) If $Q = [q_1|q_2]$, the least squares solution is $Q^T \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -11 \sqrt{6} \\ 37 \sqrt{66} \end{pmatrix}$.

20. (a) True, since if $Q$ is orthogonal, $Q^T Q = I$, so $Q^{-1} = Q^T$, which satisfies the same.

(b) True, since for any $\vec{x} \in \mathbb{R}^2$, $|Q\vec{x}|^2 = (Q\vec{x})^T(Q\vec{x}) = \vec{x}^T Q^T Q \vec{x} = \vec{x}^T \vec{x} = |\vec{x}|^2$.

23. $q_1 = a_1$, $q_2 = \frac{1}{3} a_2 - \frac{2}{3} a_1$, $q_3 = \frac{1}{5} v_3 - \frac{2}{5} v_2$. $Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, $R = \begin{pmatrix} 1 & 2 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 5 \end{pmatrix}$.

24. (a) A simple basis is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

(b) Coefficients of the hyperplane form a basis: $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$. (Or: append $[1, 0, 0, 0]$ to the matrix formed by the basis above and do GS to get the same result.)

(c) $\vec{b}_1 = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$, and $\vec{b}_2 = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$.

32. $Q_1 = I - 2 \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - 2 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. $Q_2 = I - 2 \begin{pmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -2 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$.

Note that $Q_1$ reflects in the $x$ axis, and $Q_2$ reflects in the plane $y = -z$. 4
33. The only such matrix is the identity.