1. In this problem, if the statement is always true, circle T; otherwise, circle F.

a) T F  If $A$ is a square matrix and the homogeneous equation $Ax = 0$ has only the trivial solution, then $A$ is invertible.

b) T F  If $A$ is row equivalent to $B$, then $A$ and $B$ have the same eigenvalues.

c) T F  If $A$ and $B$ have the same eigenvectors, then $A$ and $B$ have the same characteristic polynomial.

d) T F  If $A$ is diagonalizable, then $A$ has $n$ distinct eigenvalues.

e) T F  If $A$ is a matrix and $Ax = b$ has a unique solution for every $b$ in the codomain of the transformation $T(x) = Ax$, then $A$ is an invertible square matrix.

f) T F  If $A$ is an $n \times n$ matrix then $\det(-A) = -\det(A)$.

g) T F  If $A$ is an $n \times n$ matrix and its eigenvectors form a basis for $\mathbb{R}^n$, then $A$ is invertible.

h) T F  If 0 is an eigenvalue of the $n \times n$ matrix $A$, then rank$(A) < n$.

2. In this problem, if the statement is always true, circle T; if it is always false, circle F; if it is sometimes true and sometimes false, circle M.

a) T F M  If $A$ is a $3 \times 3$ matrix with characteristic polynomial $-\lambda^3 + \lambda^2 + \lambda$, then $A$ is invertible.

b) T F M  A $3 \times 3$ matrix with (only) two distinct eigenvalues is diagonalizable.

c) T F M  A diagonalizable $n \times n$ matrix admits $n$ linearly independent eigenvectors.

d) T F M  If $\det(A) = 0$, then 0 is an eigenvalue of $A$. 

3. In this problem, you need not explain your answers; just circle the correct one(s).
Let $A$ be an $n \times n$ matrix.

a) Which one of the following statements is correct?

1. An eigenvector of $A$ is a vector $v$ such that $Av = \lambda v$ for a nonzero scalar $\lambda$.

2. An eigenvector of $A$ is a nonzero vector $v$ such that $Av = \lambda v$ for a scalar $\lambda$.

3. An eigenvector of $A$ is a nonzero scalar $\lambda$ such that $Av = \lambda v$ for some vector $v$.

4. An eigenvector of $A$ is a nonzero vector $v$ such that $Av = \lambda v$ for a nonzero scalar $\lambda$.

b) Which one of the following statements is not correct?

1. An eigenvalue of $A$ is a scalar $\lambda$ such that $A - \lambda I$ is not invertible.

2. An eigenvalue of $A$ is a scalar $\lambda$ such that $(A - \lambda I)v = 0$ has a solution.

3. An eigenvalue of $A$ is a scalar $\lambda$ such that $Av = \lambda v$ for a nonzero vector $v$.

4. An eigenvalue of $A$ is a scalar $\lambda$ such that $\det(A - \lambda I) = 0$.

c) Which of the following $3 \times 3$ matrices are necessarily diagonalizable over the real numbers? (Circle all that apply.)

1. A matrix with three distinct real eigenvalues.

2. A matrix with one real eigenvalue.

3. A matrix with a real eigenvalue $\lambda$ of algebraic multiplicity 2, such that the $\lambda$-eigenspace has dimension 2.

4. A matrix with a real eigenvalue $\lambda$ such that the $\lambda$-eigenspace has dimension 2.
4. Short answer.
   a) Let $A = \begin{pmatrix} -1 & 1 \\ 1 & 7 \end{pmatrix}$, and define a transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by $T(x) = Ax$. Find the area of $T(S)$, if $S$ is a triangle in $\mathbb{R}^2$ with area 2.
   b) Suppose that $A = C \begin{pmatrix} 1/2 & 0 \\ 0 & -1 \end{pmatrix} C^{-1}$, where $C$ has columns $v_1$ and $v_2$. Given $x$ and $y$ in the picture below, draw the vectors $Ax$ and $Ay$.

![Diagram of vectors](image)

   c) Write a diagonalizable $3 \times 3$ matrix $A$ whose only eigenvalue is $\lambda = 2$.

5. Suppose we know that
   \[
   \begin{pmatrix} 4 & -10 \\ 2 & -5 \end{pmatrix} = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}^{-1}.
   \]
   Find $\begin{pmatrix} 4 & -10 \\ 2 & -5 \end{pmatrix}^{98}$.

6. Let
   \[
   A = \begin{pmatrix} 7 & 1 & 4 & 1 \\ -1 & 0 & 0 & 6 \\ 9 & 0 & 2 & 3 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 1 & 5 & 4 \\ 1 & -1 & -3 & 0 \\ -1 & 0 & 5 & 4 \\ 3 & -3 & -2 & 5 \end{pmatrix}
   \]
   a) Compute $\det(A)$.
   b) Compute $\det(B)$.
   c) Compute $\det(AB)$.
   d) Compute $\det(A^2B^{-1}AB^2)$. 
7. Give an example of a $2 \times 2$ real matrix $A$ with each of the following properties. You need not explain your answer.

a) $A$ has no real eigenvalues.

b) $A$ has eigenvalues 1 and 2.

c) $A$ is diagonalizable but not invertible.

d) $A$ is a rotation matrix with real eigenvalues.

8. Consider the matrix
$$
A = \begin{pmatrix} 4 & 2 & -4 \\ 0 & 2 & 0 \\ 2 & 2 & -2 \end{pmatrix}.
$$

a) Find the eigenvalues of $A$, and compute their algebraic multiplicities.

b) For each eigenvalue of $A$, find a basis for the corresponding eigenspace.

c) Is $A$ diagonalizable? If so, find an invertible matrix $C$ and a diagonal matrix $D$ such that $A = CDC^{-1}$. If not, why not?

9. Find all values of $a$ so that $\lambda = 1$ an eigenvalue of the matrix $A$ below.
$$
A = \begin{pmatrix} 3 & -1 & 0 & a \\ a & 2 & 0 & 4 \\ 2 & 0 & 1 & -2 \\ 13 & a & -2 & -7 \end{pmatrix}
$$

10. Consider the matrix
$$
A = \begin{pmatrix} 3\sqrt{3} -1 & -5\sqrt{3} \\ 2\sqrt{3} & -3\sqrt{3} -1 \end{pmatrix}
$$

a) Find both complex eigenvalues of $A$.

b) Find an eigenvector corresponding to each eigenvalue.