MATH 1553-C
MIDTERM EXAMINATION 2

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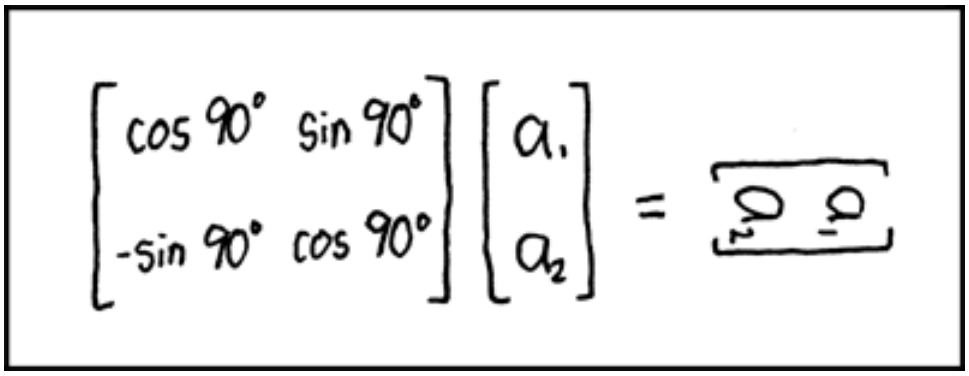
Please **read all instructions** carefully before beginning.

- Please leave your GT ID card on your desk until your TA scans your exam.
- Each problem is worth 10 points. The maximum score on this exam is 50 points.
- You have 50 minutes to complete this exam.
- There are no aids of any kind (notes, text, calculator, etc.) allowed.
- Please show your work unless instructed otherwise.
- You may cite any theorem proved in class or in the sections we covered in the text.
- Good luck!



(This is a joke, not a hint.)

Problem 1.

[2 points each]

a) Which of the following are subspaces of $\mathbf{R}^3$? Circle all that apply.

(i) the solution set of $\begin{cases} x - 2y = 0 \\ 2x - 4y = 0 \\ -x + 2y = 0 \end{cases}$ (ii) Col$\begin{pmatrix} 1 & -2 \\ 2 & -4 \\ -1 & 2 \end{pmatrix}$ (iii) Nul$\begin{pmatrix} 1 & 2 & 3 \\ -2 & -4 & -6 \end{pmatrix}$

(iv) Span$\left\{\begin{pmatrix} 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 2 \\ -4 \end{pmatrix}, \begin{pmatrix} 3 \\ -6 \end{pmatrix}\right\}$ (v) the solution set of $\begin{pmatrix} 1 & 2 & 3 \\ -2 & -4 & -6 \end{pmatrix} x = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

(vi) the range of a linear transformation with codomain $\mathbf{R}^3$

In the following questions, circle **T** if the statement is necessarily true, and circle **F** otherwise.

b) **T** **F** There exists a 3×4 matrix A such that Col$A$ is a plane and Nul$A$ is a line.

c) **T** **F** If $T : \mathbf{R}^3 \to \mathbf{R}^4$ is a one-to-one linear transformation, then the set of vectors $\{T(e_1), T(e_2), T(e_3)\}$ is a basis for the range of T.

d) **T** **F** The span of four vectors has dimension 4.

e) **T** **F** If A is an $n \times n$ matrix and $Ae_1 = Ae_2$, then Col$A$ is not equal to $\mathbf{R}^n$.

Solution.

a) (i) This is a subspace of $\mathbf{R}^2$.
(ii) This is a subspace of $\mathbf{R}^3$.
(iii) This is a subspace of $\mathbf{R}^3$.
(iv) This is a subspace of $\mathbf{R}^2$.
(v) This is a translate of the subspace in (iii), but it does not contain the zero vector, so it is not a subspace.
(vi) This is a subspace of $\mathbf{R}^3$.

b) False: the dimensions of Col$A$ and Nul$A$ sum to 4.

c) True: $T(e_1), T(e_2), T(e_3)$ are the columns of the standard matrix A for T; they span the range of T, and they are linearly independent if T is one-to-one.

d) False: the dimension is 4 if and only if the vectors are linearly independent.

e) True: if $Ae_1 = Ae_2$ then the first two columns of A are identical, so A has fewer than n pivots, so it cannot have a pivot in every row.

Problem 2. [2 points each]

In this problem, it is not necessary to show your work or justify your answers.

a) Give an example of a linear transformation $T : \mathbf{R}^3 \to \mathbf{R}^2$ that is not onto.

b) Give a basis of the xy-plane in $\mathbf{R}^3$ (the subspace consisting of all vectors with third coordinate equal to zero).

c) Write a nonzero vector in $\mathrm{Nul}A$, where $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \\ -1 & -2 \end{pmatrix}$.

d) Complete the following definition:

The dimension of a subspace V of $\mathbf{R}^n$ is ...

e) Which of the following are one-to-one transformations? (Check all that apply.)

- [x] $T : \mathbf{R}^3 \to \mathbf{R}^3$, clockwise rotation by 32° around the z-axis
- [ ] $T : \mathbf{R}^3 \to \mathbf{R}^3$, projection onto the xy-plane (i.e. set the z-coordinate to 0)
- [ ] $T : \mathbf{R}^3 \to \mathbf{R}^2$, $T(x, y, z) = (x + y, x - z)$
- [x] $T : \mathbf{R}^2 \to \mathbf{R}^3$, $T(x, y) = (x, y, y)$

Solution.

a) The zero transformation $T(x) = 0$.

b) In other words, find a pair of noncollinear vectors in the xy-plane:

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

c) In other words, find a nonzero solution of $Ax = 0$. Since the second column is twice the first, we can take $x = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

d) ... the number of vectors in any basis of V.

Problem 3.

Consider the following vectors:

$$v_1 = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} \quad v_2 = \begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix} \quad v_3 = \begin{pmatrix} 2 \\ 5 \\ -5 \end{pmatrix}$$

and the subspace $V = \text{Span}\{v_1, v_2, v_3\}$.

a) [4 points] Is $\{v_1, v_2, v_3\}$ linearly independent? If so, explain why; if not, produce an equation of linear dependence.

b) [3 points] Find a basis for V.

c) [2 points] Find another basis for V that does not include a multiple of any of the vectors in your answer for (b).

d) [1 points] What is $\dim V$?

Solution.

a) The set $\{v_1, v_2, v_3\}$ is linearly independent if and only if the matrix

$$A = \begin{pmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{pmatrix} = \begin{pmatrix} 1 & 4 & 2 \\ -2 & 1 & 5 \\ 4 & 3 & -5 \end{pmatrix}$$

has a pivot in every column. We row reduce:

$$\begin{pmatrix} 1 & 4 & 2 \\ -2 & 1 & 5 \\ 4 & 3 & -5 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

The third column does not have a pivot, so the vectors are linearly **dependent.** Finding an equation of linear dependence means finding a nontrivial solution of the homogeneous vector equation $x_1v_1 + x_2v_2 + x_3v_3 = 0$, which means putting the above system of equations into parametric vector form:

$$\begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{PVF} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}.$$

Taking $x_3 = 1$, this yields the equation of linear dependence

$$2v_1 - v_2 + v_3 = 0.$$

b) One basis for V is given by the pivot columns of A: $\{v_1, v_2\}$.below privileged amoral siren

c) We can always take the sum and difference of two linearly independent vectors to obtain another pair of linearly independent vectors: $\{v_1 + v_2, v_1 - v_2\}$.

d) Since V has a basis with two elements, its dimension is 2.

Problem 4.

In this problem we consider two linear transformations:

$T : \mathbf{R}^2 \to \mathbf{R}^2$ is defined as follows: first it flips the plane over the x-axis, then rotates the plane counterclockwise by 45°, then stretches the x-direction by a factor of 2.

$U : \mathbf{R}^2 \to \mathbf{R}^3$ is defined by the formula $U\begin{pmatrix} x \\ y \end{pmatrix} = x\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} + y\begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix}$.

a) [3 points] Compute the standard matrix for T.

b) [2 points] Compute the standard matrix for U.

c) [2 points] Compute the standard matrix for $U \circ T$.

d) [3 points] Which of the following transformations are onto? (Circle all that apply.)

$$T \qquad U \qquad U \circ T$$

Solution.

a) We follow the recipe for computing $T(e_1)$ and $T(e_2)$. Flipping over the x-axis does not move e_1; rotating counterclockwise takes e_1 to $\frac{1}{\sqrt{2}}\binom{1}{1}$; stretching the x-direction by a factor of 2 takes this to $\frac{1}{\sqrt{2}}\binom{2}{1}$. Flipping over the x-axis moves e_2 to $-e_2$; rotating counterclockwise takes $-e_2$ to $\frac{1}{\sqrt{2}}\binom{1}{-1}$; stratching the x-direction by a factor of 2 takes this to $\frac{1}{\sqrt{2}}\binom{2}{-1}$. Hence the standard matrix is

$$A = \frac{1}{\sqrt{2}}\begin{pmatrix} 2 & 2 \\ 1 & -1 \end{pmatrix}.$$

b) We could plug in $U(e_1)$ and $U(e_2)$ as above, or notice that $U\binom{x}{y}$ is defined to be the product of the matrix

$$B = \begin{pmatrix} 1 & 2 \\ 0 & 4 \\ 3 & -1 \end{pmatrix}$$

with $\binom{x}{y}$.

c) The standard matrix of $U \circ T$ is

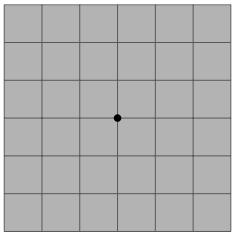
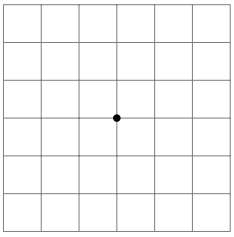
$$BA = \begin{pmatrix} 1 & 2 \\ 0 & 4 \\ 3 & -1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} 2 & 2 \\ 1 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 4 & 0 \\ 4 & -4 \\ 5 & 7 \end{pmatrix}.$$

d) Only T is onto. The other two have domain $\mathbf{R}^2$ and codomain $\mathbf{R}^3$, so it is impossible for them to be onto.

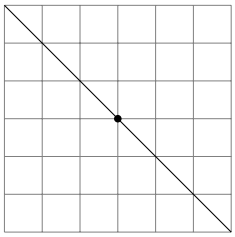
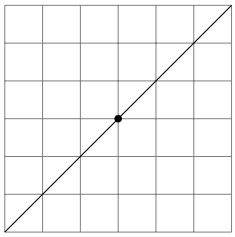
Problem 5.

[10 points]

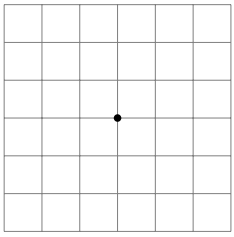
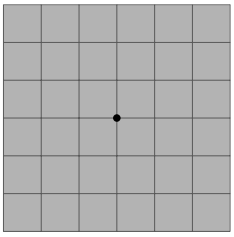
a) Write a 2×2 matrix A such that **the range of $T(x) = Ax$ is a point**, and draw pictures of $\mathrm{Nul}\,A$ and $\mathrm{Col}\,A$.

$A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ $\quad \mathrm{Nul}\,A =$  $\quad \mathrm{Col}\,A =$ 

b) Write a 2×2 matrix B such that $\mathrm{Nul}\,B$ **is a line**, and draw pictures of $\mathrm{Nul}\,B$ and $\mathrm{Col}\,B$.

$B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ $\quad \mathrm{Nul}\,B =$  $\quad \mathrm{Col}\,B =$ 

c) Write a 2×2 matrix C with **linearly independent columns**, and draw pictures of $\mathrm{Nul}\,C$ and $\mathrm{Col}\,C$.

$C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad \mathrm{Nul}\,C =$  $\quad \mathrm{Col}\,C =$ 

(In the grids, the dot is the origin.)