1. Circle T if the statement is always true, and circle F otherwise. You do not need to explain your answer.

   a) T F If \( \{v_1, v_2, v_3, v_4\} \) is a basis for a subspace \( V \) of \( \mathbb{R}^n \), then \( \{v_1, v_2, v_3\} \) is a linearly independent set.

   b) T F If \( A \) is an \( n \times n \) matrix and \( Ae_1 = Ae_2 \), then the homogeneous equation \( Ax = 0 \) has infinitely many solutions.

   c) T F The solution set of a consistent matrix equation \( Ax = b \) is a subspace.

   d) T F There exists a \( 3 \times 5 \) matrix with rank 4.

   e) T F If \( A \) is an \( 9 \times 4 \) matrix with a pivot in each column, then \( \text{Nul}A = \{0\} \).

   f) T F If \( A \) is a matrix with more rows than columns, then the transformation \( T(x) = Ax \) is not one-to-one.

   g) T F A translate of a span is a subspace.

   h) T F There exists a \( 4 \times 7 \) matrix \( A \) such that \( \text{nullity}A = 5 \).

   i) T F If \( \{v_1, v_2, \ldots, v_n\} \) is a basis for \( \mathbb{R}^4 \), then \( n = 4 \).

Solution.

a) True: if \( \{v_1, v_2, v_3\} \) is linearly dependent then \( \{v_1, v_2, v_3, v_4\} \) is automatically linearly dependent, which is impossible since \( \{v_1, v_2, v_3, v_4\} \) is a basis for a subspace.

b) True: \( x \rightarrow Ax \) is not one-to-one, so \( Ax = 0 \) has infinitely many solutions. For example, \( e_1 - e_2 \) is a non-trivial solution to \( Ax = 0 \) since \( A(e_1 - e_2) = Ae_1 - Ae_2 = 0 \).

c) False: this is true if and only if \( b = 0 \), i.e., the equation is homogeneous, in which case the solution set is the null space of \( A \).
d) **False**: the rank is the dimension of the column space, which is a subspace of $\mathbb{R}^3$, hence has dimension at most 3.

e) **True**.

f) **False**. For instance,

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

g) **False**. A subspace must contain 0.

h) **True**. For instance,

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

i) **True**. Any basis of $\mathbb{R}^4$ has 4 vectors.
2. Short answer questions: you need not explain your answers.

a) Write a nonzero vector in ColA, where $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$.

Solution.

Either column will work. For instance, $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

b) Complete the following definition:

A transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is one-to-one if…

…for every $b$ in $\mathbb{R}^m$, the equation $T(x) = b$ has at most one solution.

c) Which of the following are onto transformations? (Check all that apply.)

- ✔ $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$, reflection over the $xy$-plane
- ✗ $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$, projection onto the $xy$-plane
- ✔ $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$, project onto the $xy$-plane, forget the $z$-coordinate
- ✔ $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, scale the $x$-direction by 2

d) Let $A$ be a square matrix and let $T(x) = Ax$. Which of the following guarantee that $T$ is onto? (Check all that apply.)

- ✔ $T$ is one-to-one
- ✗ $Ax = 0$ is consistent
- ✔ Col$A = \mathbb{R}^n$
- ✔ There is a transformation $U$ such that $T \circ U(x) = x$ for all $x$
3. Parts (a) and (b) are unrelated.

a) Consider $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$T(x, y, z) = (x, x + z, 3x - 4y + z, x).$$

Is $T$ one-to-one? Justify your answer.

b) Find all real numbers $h$ so that the transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ given by

$$T(v) = \begin{pmatrix} -1 & 0 & 2 - h \\ h & 0 & 3 \end{pmatrix} v$$

is onto.

Solution.

a) One approach: We form the standard matrix $A$ for $T$:

$$A = \begin{pmatrix} T(e_1) & T(e_2) & T(e_3) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 3 & -4 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

We row-reduce $A$ until we determine its pivot columns

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 3 & -4 & 1 \\ 1 & 0 & 0 \end{pmatrix} \xrightarrow{R_3 = R_3 - 3R_1, R_4 = R_4 - R_1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -4 & 1 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -4 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

$A$ has a pivot in every column, so $T$ is one-to-one.

Alternative approach: $T$ is a linear transformation, so it is one-to-one if and only if the equation $T(x, y, z) = (0, 0, 0)$ has only the trivial solution.

If $T(x, y, z) = (x, x + z, 3x - 4y + z, x) = (0, 0, 0)$ then $x = 0$, and

$$x + z = 0 \implies 0 + z = 0 \implies z = 0, \text{ and finally}$$

$$3x - 4y + z = 0 \implies 0 - 4y + 0 = 0 \implies y = 0,$$

so the trivial solution $x = y = z = 0$ is the only solution the homogeneous equation. Therefore, $T$ is one-to-one.

b) We row-reduce $A$ to find when it will have a pivot in every row:

$$\begin{pmatrix} -1 & 0 & 2 - h \\ h & 0 & 3 \end{pmatrix} \xrightarrow{R_2 = R_2 + xR_1} \begin{pmatrix} -1 & 0 & 2 - h \\ 0 & 0 & 3 + h(2 - h) \end{pmatrix}.$$

The matrix has a pivot in every row unless

$$3 + h(2 - h) = 0, \quad h^2 - 2h - 3 = 0, \quad (h - 3)(h + 1) = 0.$$ 

Therefore, $T$ is onto as long as $h \neq 3$ and $h \neq -1.$
4.  a) Determine which of the following transformations are linear.
   (1) \( S : \mathbb{R}^2 \to \mathbb{R}^2 \) given by \( S(x_1, x_2) = (x_1, 3 + x_2) \)
   (2) \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) given by \( T(x_1, x_2) = (x_1 - x_2, x_1x_2) \)
   (3) \( U : \mathbb{R}^2 \to \mathbb{R}^3 \) given by \( U(x_1, x_2) = (-x_2, x_1, 0) \)

   b) Complete the following definition (be mathematically precise!):
      A set of vectors \( \{v_1, v_2, \ldots, v_p\} \) in \( \mathbb{R}^n \) is linearly independent if...

   c) If \( \{v_1, v_2, v_3\} \) are vectors in \( \mathbb{R}^3 \) with the property that none of the vectors is a scalar multiple of another, is \( \{v_1, v_2, v_3\} \) necessarily linearly independent? Justify your answer.

Solution.

a)  (1) \( S \) is not linear: \( S(1, 0) + (1, 0) = (2, 3) \) but \( S(1, 0) + S(1, 0) = (2, 6) \).
   (2) \( T \) is not linear: \( T(1, 1) + T(1, 1) = (0, 2), \) but \( T(2(1, 1)) = T(2, 2) = (0, 4) \).
   (3) \( U \) is linear.

b) the vector equation \( x_1v_1 + x_2v_2 + \cdots + x_pv_p = 0 \) has only the trivial solution \( x_1 = x_2 = \cdots = x_p = 0 \).

c) No. For example, take \( v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \) and \( v_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \).
   No vector in the set is a scalar multiple of any other, but nonetheless \( \{v_1, v_2, v_3\} \) is linearly dependent. In fact, \( v_3 = v_1 + v_2 \).
5. Let \( T : \mathbb{R}^3 \to \mathbb{R}^2 \) be the linear transformation which projects onto the \( yz \)-plane and then forgets the \( x \)-coordinate, and let \( U : \mathbb{R}^2 \to \mathbb{R}^2 \) be the linear transformation of rotation counterclockwise by 60°. Their standard matrices are

\[
A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \frac{1}{2} \begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix},
\]

respectively.

a) Which composition makes sense? (Circle one.)

\( U \circ T \quad T \circ U \)

b) Find the standard matrix for the transformation that you circled in (b).

**Solution.**

a) Only \( U \circ T \) makes sense, as the codomain of \( T \) is \( \mathbb{R}^2 \), which is the domain of \( U \).

b) The standard matrix for \( U \circ T \) is

\[
BA = \frac{1}{2} \begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 & 1 & -\sqrt{3} \\ 0 & \sqrt{3} & 1 \end{pmatrix}.
\]
6. Consider the following matrix $A$ and its reduced row echelon form:

$$
\begin{pmatrix}
2 & 4 & 7 & -16 \\
3 & 6 & -1 & -1 \\
5 & 10 & 6 & -17
\end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix}
1 & 2 & 0 & -1 \\
0 & 0 & 1 & -2 \\
0 & 0 & 0 & 0
\end{pmatrix}.
$$

a) Find a basis for $\text{Col} A$.

b) Find a basis $B$ for $\text{Nul} A$.

c) For each of the following vectors $v$, decide if $v$ is in $\text{Nul} A$, and if so, write $x$ as a linear combination of your basis from part (b).

$$
\begin{pmatrix}
7 \\
3 \\
1 \\
2
\end{pmatrix}
$$

Solution.

a) The pivot columns for $A$ form a basis for $\text{Col} A$, so a basis is 

$$
\left\{ \begin{pmatrix}
2 \\
3 \\
5
\end{pmatrix}, \begin{pmatrix}
7 \\
-1 \\
6
\end{pmatrix} \right\}.
$$

b) We compute the parametric vector form for the general solution of $Ax = 0$:

$$
\begin{align*}
x_1 &= -2x_2 + x_4 \\
x_2 &= x_2 \\
x_3 &= 2x_4 \\
x_4 &= x_4
\end{align*}
$$

Therefore, a basis is given by

$$
B = \left\{ \begin{pmatrix}
-2 \\
1 \\
0 \\
0
\end{pmatrix}, \begin{pmatrix}
1 \\
0 \\
2 \\
1
\end{pmatrix} \right\}
$$

c) First we note that if

$$
\begin{pmatrix}
a \\
b \\
c \\
d
\end{pmatrix} = c_1 \begin{pmatrix}
-2 \\
1 \\
0 \\
0
\end{pmatrix} + c_2 \begin{pmatrix}
1 \\
0 \\
2 \\
1
\end{pmatrix},
$$

then $c_1 = b$ and $c_2 = d$. This makes it easy to check whether a vector is in $\text{Nul} A$.

$$
\begin{pmatrix}
7 \\
3 \\
1 \\
2
\end{pmatrix} \neq 3 \begin{pmatrix}
-2 \\
1 \\
0 \\
0
\end{pmatrix} + 2 \begin{pmatrix}
1 \\
0 \\
2 \\
1
\end{pmatrix} \implies \text{not in } \text{Nul} A.
$$

$$
\begin{pmatrix}
-5 \\
-2 \\
-1
\end{pmatrix} = 2 \begin{pmatrix}
-2 \\
1 \\
0
\end{pmatrix} - \begin{pmatrix}
1 \\
0 \\
2
\end{pmatrix}.
$$
7. Consider \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) defined by

\[
T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + 2y \\ 2x + y \\ x - y \end{pmatrix}
\]

and \( U : \mathbb{R}^3 \to \mathbb{R}^2 \) defined by first projecting onto the \( xy \)-plane (forgetting the \( z \)-coordinate), then rotating counterclockwise by 90°.

a) Compute the standard matrices \( A \) and \( B \) for \( T \) and \( U \), respectively.

b) Compute the standard matrices for \( T \circ U \) and \( U \circ T \).

c) Circle all that apply:

\( T \circ U \) is: one-to-one         onto

\( U \circ T \) is: one-to-one         onto

Solution.

a) We plug in the unit coordinate vectors to get

\[
A = \begin{pmatrix} T(e_1) & T(e_2) \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}
\]

and

\[
B = \begin{pmatrix} U(e_1) & U(e_2) & U(e_3) \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.
\]

b) The standard matrix for \( T \circ U \) is

\[
AB = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 0 \\ 1 & -2 & 0 \\ -1 & -1 & 0 \end{pmatrix}.
\]

The standard matrix for \( U \circ T \) is

\[
BA = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} -2 & -1 \\ -1 & -2 \\ 1 & 2 \end{pmatrix}.
\]

c) Looking at the matrices, we see that \( T \circ U \) is not one-to-one or onto, and that \( U \circ T \) is one-to-one and onto.
8.  

a) Write a $2 \times 2$ matrix $A$ with rank 2, and draw pictures of $\text{Nul} A$ and $\text{Col} A$.

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$  

$\text{Nul} A =$  

$\text{Col} A =$

b) Write a $2 \times 2$ matrix $B$ with rank 1, and draw pictures of $\text{Nul} B$ and $\text{Col} B$.

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$  

$\text{Nul} B =$  

$\text{Col} B =$

c) Write a $2 \times 2$ matrix $C$ with rank 0, and draw pictures of $\text{Nul} C$ and $\text{Col} C$.

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$  

$\text{Nul} C =$  

$\text{Col} C =$

(In the grids, the dot is the origin.)