MATH 1553-C
MIDTERM EXAMINATION 1

Name | GT Email | @gatech.edu

Please read all instructions carefully before beginning.

- Please leave your GT ID card on your desk until your TA scans your exam.
- Each problem is worth 10 points. The maximum score on this exam is 50 points.
- You have 50 minutes to complete this exam.
- There are no aids of any kind (notes, text, calculator, etc.) allowed.
- Please show your work.
- You may cite any theorem proved in class or in the sections we covered in the text.
- Good luck!
Problem 1. [2 points each]

Parts (a) and (b) refer to the following matrix:

\[
A = \begin{pmatrix} 1 & -2 \\ 2 & -4 \\ -1 & 2 \end{pmatrix}.
\]

a) What is the best way to describe the span of the columns \( A \)?

- a line in \( \mathbb{R}^2 \)
- a line in \( \mathbb{R}^3 \)
- a plane in \( \mathbb{R}^2 \)
- a plane in \( \mathbb{R}^3 \)

b) What is the best way to describe the solution set of \( Ax = 0 \)?

- a line in \( \mathbb{R}^2 \)
- a line in \( \mathbb{R}^3 \)
- a plane in \( \mathbb{R}^2 \)
- a plane in \( \mathbb{R}^3 \)

In the following questions, circle \( T \) if the statement is necessarily true, and circle \( F \) otherwise.

c) \(\text{T}\)\(\text{ F}\) The following matrix is in row echelon form:

\[
\begin{pmatrix} 1 & 7 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.
\]

d) \(\text{T}\)\(\text{ F}\) If \( A \) is a \( 2 \times 3 \) matrix, then \( Ax = b \) can have a unique solution.

e) \(\text{T}\)\(\text{ F}\) If \( A \) is an \( m \times n \) matrix, then the solution set of \( Ax = b \) is empty or it is a span in \( \mathbb{R}^n \).

Solution.

a) The columns are collinear nonzero vectors in \( \mathbb{R}^3 \), so they span a line in \( \mathbb{R}^3 \).

b) The reduced row echelon form of \( A \) is

\[
\begin{pmatrix} 1 & -2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.
\]

There are two variables, one of which is free, so the solution set is a line in \( \mathbb{R}^2 \).

c) True.

d) False. Since \( A \) has more columns than rows, one variable will always be free.

e) False. It could be a translate of a span in \( \mathbb{R}^n \).
Problem 2. [2 points each]

In this problem, it is not necessary to show your work or justify your answers.

a) Compute the product (your answer will be a single vector):

\[
\begin{pmatrix}
1 & 3 & -1 \\
2 & 0 & 4
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix} =
\]

b) Give an example of a $3 \times 4$ matrix $A$ such that $Ax = b$ is consistent for all $b$ in $\mathbb{R}^3$.

c) Find a matrix $A$ with three rows such that $Ax = b$ is consistent if and only if $b$ is a linear combination of the vectors

\[
\begin{pmatrix}
1 \\
7 \\
-3
\end{pmatrix}
\text{ and }
\begin{pmatrix}
0 \\
1 \\
4
\end{pmatrix}.
\]

d) Find three vectors $u, v, w$ in $\mathbb{R}^3$ whose span is

\[
\{(x, y, 0) \mid x, y \text{ are in } \mathbb{R}\} \quad \text{(the } xy\text{-plane)}.
\]

e) If $A$ is a $2 \times 2$ matrix such that $A\begin{pmatrix}1 \\ 0 \end{pmatrix} = \begin{pmatrix}-1 \\ 2 \end{pmatrix}$ and $A\begin{pmatrix}0 \\ 1 \end{pmatrix} = \begin{pmatrix}3 \\ 3 \end{pmatrix}$, then what is $A\begin{pmatrix}1 \\ -2 \end{pmatrix}$?

Solution.

a) \[
\begin{pmatrix}
x + 3y - z \\
2x + 4z
\end{pmatrix}
\]

b) We must write down a matrix with a pivot in each row. Here is one:

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix}.
\]

c) The matrix equation $Ax = b$ is consistent if and only if $b$ is in the span of the columns of $A$. Hence we may take

\[
A = \begin{pmatrix}
1 & 0 \\
7 & 1 \\
-3 & 4
\end{pmatrix}.
\]

d) There are many correct answers. One is:

\[
u = \begin{pmatrix}1 \\ 0 \\ 0\end{pmatrix} \quad v = \begin{pmatrix}0 \\ 1 \\ 0\end{pmatrix} \quad w = \begin{pmatrix}0 \\ 0 \\ 0\end{pmatrix}.
\]

e) $A\begin{pmatrix}1 \\ -2 \end{pmatrix} = A\left(\begin{pmatrix}1 \\ 0 \end{pmatrix} - 2\begin{pmatrix}0 \\ 1 \end{pmatrix}\right) = A\begin{pmatrix}1 \\ 0 \end{pmatrix} - 2A\begin{pmatrix}0 \\ 1 \end{pmatrix} = \begin{pmatrix}-1 \\ 2 \end{pmatrix} - 2\begin{pmatrix}3 \\ 3 \end{pmatrix} = \begin{pmatrix}-7 \\ -4 \end{pmatrix}$. 


Problem 3.

Consider the following system of linear equations:

\[-2x_1 + x_2 - 4x_3 + x_4 = 4\]
\[-x_1 + x_2 - x_3 = 1\]
\[x_2 + 2x_3 - x_4 = -2\]

a) [2 points] Write the system as a vector equation.

b) [2 points] Write the system as a matrix equation.

c) [1 point ] Write the system as an augmented matrix.

d) [2 points] Row-reduce this matrix to reduced row echelon form.

e) [2 points] Write the parametric vector form of the general solution.

f) [1 point ] Write any solution of the original equation.

Solution.

a) \[x_1 \begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + x_3 \begin{pmatrix} -4 \\ -1 \\ 2 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix}\]

b) \[
\begin{pmatrix}
-2 & 1 & -4 & 1 \\
-1 & 1 & -1 & 0 \\
0 & 1 & 2 & -1
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{pmatrix} =
\begin{pmatrix}
4 \\
1 \\
-2
\end{pmatrix}
\]

c) \[
\begin{pmatrix}
-2 & 1 & -4 & 1 & 4 \\
-1 & 1 & -1 & 0 & 1 \\
0 & 1 & 2 & -1 & -2
\end{pmatrix}
\]
d) We row reduce the augmented matrix:

\[
\begin{pmatrix}
-2 & 1 & -4 & 1 & | & 4 \\
-1 & 1 & -1 & 0 & | & 1 \\
0 & 1 & 2 & -1 & | & -2 \\
\end{pmatrix}
\]

\[
\rightarrow
\begin{pmatrix}
1 & -1 & 1 & 0 & | & -1 \\
-2 & 1 & -4 & 1 & | & 4 \\
0 & 1 & 2 & -1 & | & -2 \\
\end{pmatrix}
\]

\[
R_1 \rightarrow R_1 \rightarrow
\begin{pmatrix}
1 & -1 & 1 & 0 & | & -1 \\
-2 & 1 & -4 & 1 & | & 4 \\
0 & 1 & 2 & -1 & | & -2 \\
\end{pmatrix}
\]

\[
R_2 = R_2 + 2R_1 \rightarrow
\begin{pmatrix}
1 & -1 & 1 & 0 & | & -1 \\
0 & -1 & -2 & 1 & | & 2 \\
0 & 1 & 2 & -1 & | & -2 \\
\end{pmatrix}
\]

\[
R_3 = R_3 - R_2 \rightarrow
\begin{pmatrix}
1 & -1 & 1 & 0 & | & -1 \\
0 & 1 & 2 & -1 & | & -2 \\
0 & 1 & 2 & -1 & | & -2 \\
\end{pmatrix}
\]

\[
R_4 = R_4 - R_3 \rightarrow
\begin{pmatrix}
1 & 0 & 3 & -1 & | & -3 \\
0 & 1 & 2 & -1 & | & -2 \\
0 & 0 & 0 & 0 & | & 0 \\
\end{pmatrix}
\]

This corresponds to the system of equations:

\[
x_1 + 3x_3 - x_4 = -3 \\
x_2 + 2x_3 - x_4 = -2.
\]

The free variables are \(x_3\) and \(x_4\), so the parametric form is

\[
x_1 = -3x_3 + x_4 - 3 \\
x_2 = -2x_3 + x_4 - 2 \\
x_3 = x_3 \\
x_4 = x_4.
\]

The parametric vector form is then

\[
\begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
\end{pmatrix} = x_3 \begin{pmatrix}
-3 \\
-2 \\
1 \\
0 \\
\end{pmatrix} + x_4 \begin{pmatrix}
1 \\
1 \\
0 \\
1 \\
\end{pmatrix} + \begin{pmatrix}
-3 \\
2 \\
0 \\
0 \\
\end{pmatrix}.
\]

e) Taking \(x_3 = x_4 = 0\), one solution is \(\begin{pmatrix}
-3 \\
2 \\
0 \\
0 \\
\end{pmatrix}\).
Problem 4.

In this problem, we consider the matrix

\[ A = \begin{pmatrix} 1 & 2 \\ -2 & -4 \end{pmatrix}. \]

On the grid on the left, draw and label:

a) [3 points] The solution set of \( Ax = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \). If the system is inconsistent, write “inconsistent”.

b) [2 points] The solution set of \( Ax = 0 \). If the system is inconsistent, write “inconsistent”.

On the grid on the right, draw and label:

c) [3 points] The span of the columns of \( A \).

d) [2 points] A vector \( b \) such that \( Ax = b \) is inconsistent. If no such \( b \) exists, explain why.

Solution.

a) We form an augmented matrix and row reduce:

\[
\begin{pmatrix} 1 & 2 & -1 \\ -2 & -4 & 2 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{\text{PF}} x = -2y - 1 \quad \xrightarrow{\text{PVF}} \begin{pmatrix} x \\ y \end{pmatrix} = y \begin{pmatrix} -2 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix}.
\]

This is the line through \( \begin{pmatrix} -1 \\ 2 \end{pmatrix} \) parallel to the vector \( \begin{pmatrix} -2 \\ 1 \end{pmatrix} \).

b) The solution set of the corresponding homogeneous system is the parallel line through the origin.

c) The columns are collinear, so their span is a line.

d) We can take any vector \( b \) not lying on the column span.
Problem 5.

a) [6 points] Find all values of $k$ such that
\[
\begin{pmatrix}
1 \\
k \\
6
\end{pmatrix}
\text{ is in } \text{Span}\left\{ \begin{pmatrix}2 \\ 0 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix}1 \\ -1 \\ 2 \end{pmatrix} \right\}.
\]

b) [4 points] For every number $k$ that you found in (a), express \( \begin{pmatrix}1 \\ k \\ 6 \end{pmatrix} \) as a linear combination of \( \begin{pmatrix}2 \\ 0 \\ 4 \end{pmatrix} \) and \( \begin{pmatrix}-1 \\ 3 \\ 2 \end{pmatrix} \).

Solution.

a) We are asking when the vector equation
\[
x \begin{pmatrix}2 \\ 0 \\ 4 \end{pmatrix} + y \begin{pmatrix}-1 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix}1 \\ k \\ 6 \end{pmatrix}
\]
is consistent. We form an augmented matrix and row reduce:
\[
\begin{pmatrix}
2 & -1 & 1 \\
0 & 3 & k \\
4 & 2 & 6
\end{pmatrix}
\rightarrow
\begin{pmatrix}
2 & -1 & 1 \\
0 & 3 & k \\
4 & 2 & 6
\end{pmatrix}
\rightarrow
\begin{pmatrix}
2 & -1 & 1 \\
0 & 3 & k \\
0 & 4 & 4
\end{pmatrix}
\rightarrow
\begin{pmatrix}
2 & -1 & 1 \\
0 & 3 & k \\
0 & 4 & 4
\end{pmatrix}
\rightarrow
\begin{pmatrix}
2 & -1 & 1 \\
0 & 3 & k \\
0 & 0 & k - 3
\end{pmatrix}.
\]

This is consistent if and only if $k = 3$.

b) When $k = 3$, we have $y = 1$ and $2x - y = 1$, which implies $x = 1$. Hence
\[
\begin{pmatrix}1 \\ 3 \\ 6 \end{pmatrix} = 1 \cdot \begin{pmatrix}2 \\ 0 \\ 4 \end{pmatrix} + 1 \cdot \begin{pmatrix}-1 \\ 3 \\ 2 \end{pmatrix}.
\]