MATH 1553-C
MIDTERM EXAMINATION 1

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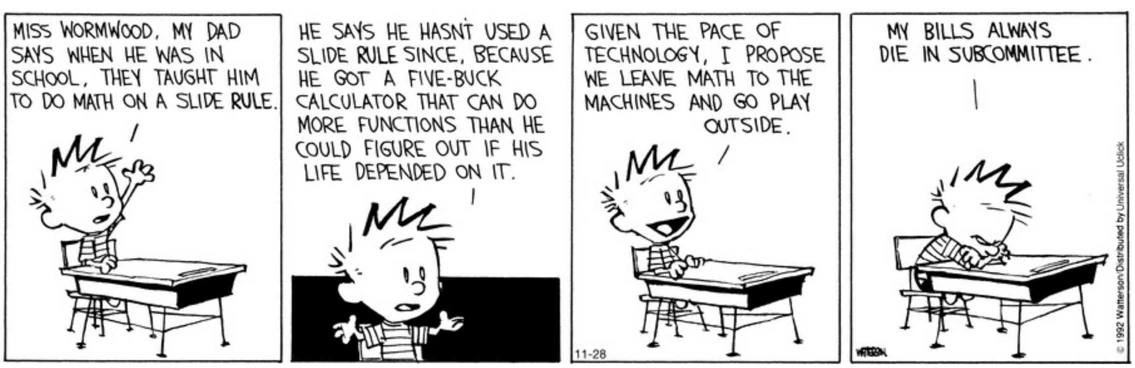
Please **read all instructions** carefully before beginning.

- Please leave your GT ID card on your desk until your TA scans your exam.
- Each problem is worth 10 points. The maximum score on this exam is 50 points.
- You have 50 minutes to complete this exam.
- There are no aids of any kind (notes, text, calculator, etc.) allowed.
- Please show your work.
- You may cite any theorem proved in class or in the sections we covered in the text.
- Good luck!



Problem 1. [2 points each]

Parts (a) and (b) refer to the following matrix:

$$A = \begin{pmatrix} 1 & -2 \\ 2 & -4 \\ -1 & 2 \end{pmatrix}.$$

a) What is the best way to describe the span of the columns A?

a line in $\mathbf{R}^2$ a line in $\mathbf{R}^3$ a plane in $\mathbf{R}^2$ a plane in $\mathbf{R}^3$

b) What is the best way to describe the solution set of $Ax = 0$?

a line in $\mathbf{R}^2$ a line in $\mathbf{R}^3$ a plane in $\mathbf{R}^2$ a plane in $\mathbf{R}^3$

In the following questions, circle **T** if the statement is necessarily true, and circle **F** otherwise.

c) **T** **F** The following matrix is in row echelon form:

$$\left(\begin{array}{ccc|c} 1 & 7 & 2 & 4 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 15 \end{array}\right)$$

d) **T** **F** If A is a 2×3 matrix, then $Ax = b$ can have a unique solution.

e) **T** **F** If A is an $m \times n$ matrix, then the solution set of $Ax = b$ is empty or it is a span in $\mathbf{R}^n$.

Solution.

a) The columns are collinear nonzero vectors in $\mathbf{R}^3$, so they span a line in $\mathbf{R}^3$.

b) The reduced row echelon form of A is

$$\begin{pmatrix} 1 & -2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

There are two variables, one of which is free, so the solution set is a line in $\mathbf{R}^2$.

c) **True.**

d) **False.** Since A has more columns than rows, one variable will always be free.

e) **False.** It could be a *translate* of a span in $\mathbf{R}^n$.

Problem 2.

[2 points each]

In this problem, it is not necessary to show your work or justify your answers.

a) Compute the product (your answer will be a single vector):

$$\begin{pmatrix} 1 & 3 & -1 \\ 2 & 0 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} =$$

b) Give an example of a 3×4 matrix A such that $Ax = b$ is consistent for all b in $\mathbf{R}^3$.

c) Find a matrix A with three rows such that $Ax = b$ is consistent if and only if b is a linear combination of the vectors

$$\begin{pmatrix} 1 \\ 7 \\ -3 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix}.$$

d) Find three vectors u, v, w in $\mathbf{R}^3$ whose span is

$$\big\{(x, y, 0) \mid x, y \text{ are in } \mathbf{R}\big\} \qquad \text{(the } xy\text{-plane).}$$

e) If A is a 2×2 matrix such that $A\binom{1}{0} = \binom{-1}{2}$ and $A\binom{0}{1} = \binom{3}{3}$, then what is $A\binom{1}{-2}$?

Solution.

a) $\begin{pmatrix} x + 3y - z \\ 2x + 4z \end{pmatrix}$

b) We must write down a matrix with a pivot in each row. Here is one:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

c) The matrix equation $Ax = b$ is consistent if and only if b is in the span of the columns of A. Hence we may take

$$A = \begin{pmatrix} 1 & 0 \\ 7 & 1 \\ -3 & 4 \end{pmatrix}.$$

d) There are many correct answers. One is:

$$u = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad v = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad w = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

e) $A\begin{pmatrix} 1 \\ -2 \end{pmatrix} = A\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} - 2\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = A\begin{pmatrix} 1 \\ 0 \end{pmatrix} - 2A\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} - 2\begin{pmatrix} 3 \\ 3 \end{pmatrix} = \begin{pmatrix} -7 \\ -4 \end{pmatrix}.$

Problem 3.

Consider the following system of linear equations:

$$\begin{aligned} -2x_1 + x_2 - 4x_3 + x_4 &= 4 \\ -x_1 + x_2 - x_3 &= 1 \\ x_2 + 2x_3 - x_4 &= -2 \end{aligned}$$

a) [2 points] Write the system as a vector equation.

b) [2 points] Write the system as a matrix equation.

c) [1 point] Write the system as an augmented matrix.

d) [2 points] Row-reduce this matrix to reduced row echelon form.

e) [2 points] Write the parametric vector form of the general solution.

f) [1 point] Write any solution of the original equation.

Solution.

a) $x_1 \begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + x_3 \begin{pmatrix} -4 \\ -1 \\ 2 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix}$

b) $\begin{pmatrix} -2 & 1 & -4 & 1 \\ -1 & 1 & -1 & 0 \\ 0 & 1 & 2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix}$

c) $\left(\begin{array}{cccc|c} -2 & 1 & -4 & 1 & 4 \\ -1 & 1 & -1 & 0 & 1 \\ 0 & 1 & 2 & -1 & -2 \end{array}\right)$

d) We row reduce the augmented matrix:

$$\left(\begin{array}{cccc|c} -2 & 1 & -4 & 1 & 4 \\ -1 & 1 & -1 & 0 & 1 \\ 0 & 1 & 2 & -1 & -2 \end{array}\right) \xrightarrow{R_1 \longleftrightarrow R_2} \left(\begin{array}{cccc|c} -1 & 1 & -1 & 0 & 1 \\ -2 & 1 & -4 & 1 & 4 \\ 0 & 1 & 2 & -1 & -2 \end{array}\right)$$

$$\xrightarrow{R_1 = -R_1} \left(\begin{array}{cccc|c} 1 & -1 & 1 & 0 & -1 \\ -2 & 1 & -4 & 1 & 4 \\ 0 & 1 & 2 & -1 & -2 \end{array}\right)$$

$$\xrightarrow{R_2 = R_2 + 2R_1} \left(\begin{array}{cccc|c} 1 & -1 & 1 & 0 & -1 \\ 0 & -1 & -2 & 1 & 2 \\ 0 & 1 & 2 & -1 & -2 \end{array}\right)$$

$$\xrightarrow{R_2 = -R_2} \left(\begin{array}{cccc|c} 1 & -1 & 1 & 0 & -1 \\ 0 & 1 & 2 & -1 & -2 \\ 0 & 1 & 2 & -1 & -2 \end{array}\right)$$

$$\xrightarrow[R_1 = R_1 + R_2]{R_3 = R_3 - R_2} \left(\begin{array}{cccc|c} 1 & 0 & 3 & -1 & -3 \\ 0 & 1 & 2 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

This corresponds to the system of equations:

$$\begin{aligned} x_1 \qquad + 3x_3 - x_4 &= -3 \\ x_2 + 2x_3 - x_4 &= -2. \end{aligned}$$

The free variables are x_3 and x_4, so the parametric form is

$$\begin{aligned} x_1 &= -3x_3 + x_4 - 3 \\ x_2 &= -2x_3 + x_4 - 2 \\ x_3 &= x_3 \\ x_4 &= x_4. \end{aligned}$$

The parametric vector form is then

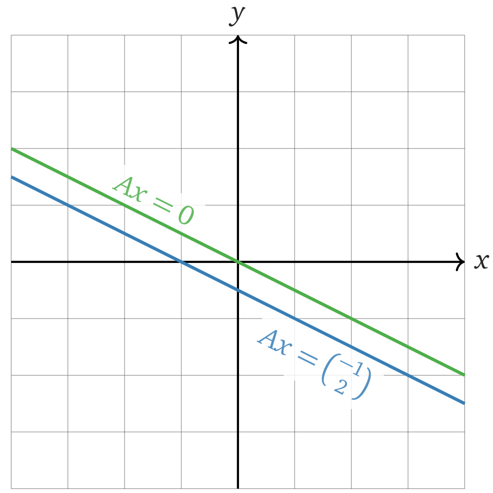
$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} -3 \\ -2 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -3 \\ 2 \\ 0 \\ 0 \end{pmatrix}.$$

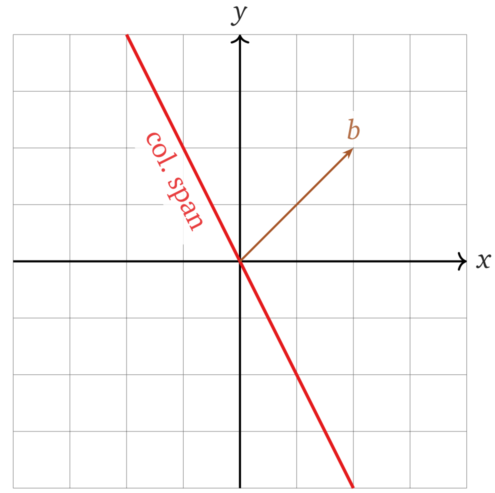
e) Taking $x_3 = x_4 = 0$, one solution is $\begin{pmatrix} -3 \\ 2 \\ 0 \\ 0 \end{pmatrix}$.

Problem 4.

In this problem, we consider the matrix

$$A = \begin{pmatrix} 1 & 2 \\ -2 & -4 \end{pmatrix}.$$





On the grid on the **left**, draw and label:

a) [3 points] The solution set of $Ax = \binom{-1}{2}$. If the system is inconsistent, write "inconsistent".

b) [2 points] The solution set of $Ax = 0$. If the system is inconsistent, write "inconsistent".

On the grid on the **right**, draw and label:

c) [3 points] The span of the columns of A.

d) [2 points] A vector b such that $Ax = b$ is inconsistent. If no such b exists, explain why.

Solution.

a) We form an augmented matrix and row reduce:

$$\left(\begin{array}{cc|c} 1 & 2 & -1 \\ -2 & -4 & 2 \end{array}\right) \xrightarrow{\text{RREF}} \left(\begin{array}{cc|c} 1 & 2 & -1 \\ 0 & 0 & 0 \end{array}\right) \xrightarrow{\text{PF}} \begin{array}{rl} x = & -2y - 1 \\ y = & y \end{array} \xrightarrow{\text{PVF}} \begin{pmatrix} x \\ y \end{pmatrix} = y\begin{pmatrix} -2 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

This is the line through $\binom{-1}{0}$ parallel to the vector $\binom{-2}{1}$.

b) The solution set of the corresponding homogeneous system is the parallel line through the origin.

c) The columns are collinear, so their span is a line.

d) We can take any vector b not lying on the column span.

Problem 5.

a) [6 points] Find all values of k such that

$$\begin{pmatrix}1\\k\\6\end{pmatrix} \quad \text{is in} \quad \text{Span}\left\{\begin{pmatrix}2\\0\\4\end{pmatrix}, \begin{pmatrix}-1\\3\\2\end{pmatrix}\right\}.$$

b) [4 points] For every number k that you found in (a), express $\begin{pmatrix}1\\k\\6\end{pmatrix}$ as a linear combination of $\begin{pmatrix}2\\0\\4\end{pmatrix}$ and $\begin{pmatrix}-1\\3\\2\end{pmatrix}$.

Solution.

a) We are asking when the vector equation

$$x\begin{pmatrix}2\\0\\4\end{pmatrix} + y\begin{pmatrix}-1\\3\\2\end{pmatrix} = \begin{pmatrix}1\\k\\6\end{pmatrix}$$

is consistent. We form an augmented matrix and row reduce:

$$\left(\begin{array}{cc|c}2 & -1 & 1\\0 & 3 & k\\4 & 2 & 6\end{array}\right) \xrightarrow{R_3=R_3-2R_1} \left(\begin{array}{cc|c}2 & -1 & 1\\0 & 3 & k\\0 & 4 & 4\end{array}\right) \xrightarrow{R_2\longleftrightarrow R_3} \left(\begin{array}{cc|c}2 & -1 & 1\\0 & 4 & 4\\0 & 3 & k\end{array}\right)$$

$$\xrightarrow{R_2=R_2\div 4} \left(\begin{array}{cc|c}2 & -1 & 1\\0 & 1 & 1\\0 & 3 & k\end{array}\right) \xrightarrow{R_3=R_3-3R_2} \left(\begin{array}{cc|c}2 & -1 & 1\\0 & 1 & 1\\0 & 0 & k-3\end{array}\right).$$

This is consistent if and only if $k = 3$.

b) When $k = 3$, we have $y = 1$ and $2x - y = 1$, which implies $x = 1$. Hence

$$\begin{pmatrix}1\\3\\6\end{pmatrix} = 1\cdot\begin{pmatrix}2\\0\\4\end{pmatrix} + 1\cdot\begin{pmatrix}-1\\3\\2\end{pmatrix}.$$