For those who want additional practice problems after completing the worksheet, here are some extra practice problems.

1. a) If \( A \) is the matrix that implements rotation by \( 143^\circ \) in \( \mathbb{R}^2 \), then \( A \) has no real eigenvalues.

b) If \( A \) is diagonalizable and invertible, then \( A^{-1} \) is diagonalizable.

c) A \( 3 \times 3 \) (real) matrix can have eigenvalues \( 3, 5, \) and \( 2 + i \).

Solution.

a) True. If \( A \) had a real eigenvalue \( \lambda \), then we would have \( Ax = \lambda x \) for some vector \( x \) in \( \mathbb{R}^2 \). This means that \( x \) would lie on the same line through the origin as the rotation of \( x \) by \( 143^\circ \), which is impossible.

b) True. If \( A = CDC^{-1} \) and \( A \) is invertible then its eigenvalues are all nonzero, so the diagonal entries of \( D \) are nonzero and thus \( D \) is invertible (pivot in every diagonal position). Thus, \( A^{-1} = (CDC^{-1})^{-1} = (C^{-1})^{-1}D^{-1}C^{-1} = CD^{-1}C^{-1} \).

c) False. If \( 2 + i \) is an eigenvalue then so is its conjugate \( 2 - i \).

2. Let \( A = \begin{pmatrix} 8 & 36 & 62 \\ -6 & -34 & -62 \\ 3 & 18 & 33 \end{pmatrix} \).

The characteristic polynomial for \( A \) is \( -\lambda^3 + 7\lambda^2 - 16\lambda + 12 \), and \( \lambda - 3 \) is a factor.

Decide if \( A \) is diagonalizable. If it is, find an invertible matrix \( C \) and a diagonal matrix \( D \) such that \( A = CDC^{-1} \).

Solution.

By polynomial division,

\[
\frac{-\lambda^3 + 7\lambda^2 - 16\lambda + 12}{\lambda - 3} = -\lambda^2 + 4\lambda - 4 = -(\lambda - 2)^2.
\]

Thus, the characteristic poly factors as \( -(\lambda - 3)(\lambda - 2)^2 \), so the eigenvalues are \( \lambda_1 = 3 \) and \( \lambda_2 = 2 \).

For \( \lambda_1 = 3 \), we row-reduce \( A - 3I \):

\[
\begin{pmatrix} 5 & 36 & 62 \\ -6 & -37 & -62 \\ 3 & 18 & 30 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{pmatrix} 1 & 6 & 10 \\ -6 & -37 & -62 \\ 5 & 36 & 62 \end{pmatrix} \xrightarrow{R_2 = R_2 + 6R_1} \begin{pmatrix} 1 & 6 & 10 \\ 0 & -1 & -2 \\ 0 & 6 & 12 \end{pmatrix} \xrightarrow{R_3 = R_3 - 5R_1} \begin{pmatrix} 1 & 6 & 10 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_3 = R_3 + 6R_2} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}.
\]
Therefore, the solutions to \((A - 3I \mid 0)\) are \(x_1 = 2x_3, \ x_2 = -2x_3, \ x_3 = x_3\).

\[
\begin{pmatrix}
    x_1 \\
    x_2 \\
    x_3
\end{pmatrix} = \begin{pmatrix}
    2x_3 \\
    -2x_3 \\
    x_3
\end{pmatrix} = x_3 \begin{pmatrix}
    2 \\
    -2 \\
    1
\end{pmatrix}.
\]

The 3-eigenspace has basis \(\left\{ \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \right\}\).

For \(\lambda = 2\), we row-reduce \(A - 2I\):}

\[
\begin{pmatrix}
    6 & 36 & 62 \\
    -6 & -36 & -62 \\
    3 & 18 & 31
\end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix}
    1 & 6 & 31/3 \\
    0 & 0 & 0 \\
    0 & 0 & 0
\end{pmatrix}.
\]

The solutions to \((A - 2I \mid 0)\) are \(x_1 = -6x_2 - \frac{31}{3}x_3, \ x_2 = x_2, \ x_3 = x_3\).

\[
\begin{pmatrix}
    x_1 \\
    x_2 \\
    x_3
\end{pmatrix} = \begin{pmatrix}
    -6x_2 - \frac{31}{3}x_3 \\
    x_2 \\
    x_3
\end{pmatrix} = x_2 \begin{pmatrix}
    -6 \\
    1 \\
    0
\end{pmatrix} + x_3 \begin{pmatrix}
    -\frac{31}{3} \\
    0 \\
    1
\end{pmatrix}.
\]

The 2-eigenspace has basis \(\left\{ \begin{pmatrix} -6 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{31}{3} \\ 0 \\ 1 \end{pmatrix} \right\}\).

Therefore, \(A = CDC^{-1}\) where

\[
C = \begin{pmatrix}
    2 & -6 & -\frac{31}{3} \\
    -2 & 1 & 0 \\
    1 & 0 & 1
\end{pmatrix} \quad D = \begin{pmatrix}
    3 & 0 & 0 \\
    0 & 2 & 0 \\
    0 & 0 & 2
\end{pmatrix}.
\]

Note that we arranged the eigenvectors in \(C\) in order of the eigenvalues \(3, 2, 2\), so we had to put the diagonals of \(D\) in the same order.

3. Give examples of \(2 \times 2\) matrices with the following properties. Justify your answers.
   
   a) A matrix \(A\) which is invertible and diagonalizable.
   
   b) A matrix \(B\) which is invertible but not diagonalizable.
   
   c) A matrix \(C\) which is not invertible but is diagonalizable.
   
   d) A matrix \(D\) which is neither invertible nor diagonalizable.

Solution.

   a) We can take any diagonal matrix with nonzero diagonal entries:

\[
A = \begin{pmatrix}
    1 & 0 \\
    0 & 1
\end{pmatrix}.
\]

   b) A shear has only one eigenvalue \(\lambda = 1\). The associated eigenspace is the \(x\)-axis, so there do not exist two linearly independent eigenvectors. Hence it is not diagonalizable.

\[
B = \begin{pmatrix}
    1 & 1 \\
    0 & 1
\end{pmatrix}.
\]
c) We can take any diagonal matrix with some zero diagonal entries:
\[ C = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \]

d) Such a matrix can only have the eigenvalue zero — otherwise it would have two eigenvalues, hence be diagonalizable. Thus the characteristic polynomial is \( f(\lambda) = \lambda^2 \). Here is a matrix with trace and determinant zero, whose zero-eigenspace (i.e., null space) is not all of \( \mathbb{R}^2 \):
\[ D = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \]

4. Let \( A = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \). Find all eigenvalues of \( A \). For each eigenvalue, find an associated eigenvector.

**Solution.**

The characteristic polynomial is

\[
\lambda^2 - \text{Tr}(A)\lambda + \det(A) = \lambda^2 - 2\lambda + 5
\]

\[
\lambda^2 - 2\lambda + 5 = 0 \iff \lambda = \frac{2 \pm \sqrt{4 - 20}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i.
\]

For \( \lambda_1 = 1 - 2i \), we find an eigenvector. For \( A - (1 - 2i)I \), the second row must be a multiple of the first since \( A - (1 - 2i)I \) is a non-invertible \( 2 \times 2 \) matrix, so row-reduction will automatically destroy the second row.

\[
\begin{pmatrix} A - (1 - 2i)I & 0 \end{pmatrix} = \begin{pmatrix} 2i & 2 \\ -2 & 2i \end{pmatrix} \xrightarrow{R_2 = R_2 - iR_1 \text{ then } R_1 = R_1/(2i)} \begin{pmatrix} 1 & -i \\ 0 & 0 \end{pmatrix}.
\]

So \( x_1 = ix_2 \) and \( x_2 \) is free. An eigenvector is \( \mathbf{v}_1 = \begin{pmatrix} i \\ 1 \end{pmatrix} \).

An eigenvector for \( \lambda_2 = 1 + i \) is \( \mathbf{v}_2 = \overline{\mathbf{v}_1} = \begin{pmatrix} -i \\ 1 \end{pmatrix} \).

Alternatively: for the eigenvalue \( \lambda = 1 - 2i \), we can use a trick you may have seen in class: the first row \( \begin{pmatrix} a & b \end{pmatrix} \) of \( A - \lambda I \) will lead to an eigenvector \( \begin{pmatrix} -b \\ a \end{pmatrix} \) (or equivalently, \( \begin{pmatrix} b \\ -a \end{pmatrix} \) if you prefer).

\[
\begin{pmatrix} A - (1 - 2i)I & 0 \end{pmatrix} = \begin{pmatrix} 2i \\ (\ast) \end{pmatrix} \xrightarrow{(\ast)} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies \mathbf{v} = \begin{pmatrix} -2 \\ 2i \end{pmatrix}.
\]

Note that this choice of \( \mathbf{v} \) looks much different than the vector \( \mathbf{v}_1 \) above, but they are actually equivalent since they are (complex) scalar multiples of each other, as \( \mathbf{v} = 2i\mathbf{v}_1 \). From the correspondence between conjugate eigenvalues and their
eigenvectors, we know (without doing any additional work!) that for the eigenvalue \( \lambda = 1 + 2i \), a corresponding eigenvector is \( w = \bar{v} = \begin{pmatrix} -2 \\ -2i \end{pmatrix} \).

5. Suppose a \( 2 \times 2 \) matrix \( A \) has eigenvalue \( \lambda_1 = -2 \) with eigenvector \( v_1 = \begin{pmatrix} 3/2 \\ 1 \end{pmatrix} \), and eigenvalue \( \lambda_2 = -1 \) with eigenvector \( v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \).

a) Find \( A \).

b) Find \( A^{100} \).

**Solution.**

a) We have \( A = CDC^{-1} \) where

\[
C = \begin{pmatrix} 3/2 & 1 \\ 1 & -1 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}.
\]

We compute \( C^{-1} = \frac{1}{-5/2} \begin{pmatrix} -1 & -1 \\ -1 & 3/2 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 2 & 2 \\ 2 & -3 \end{pmatrix} \).

\[
A = CDC^{-1} = \frac{1}{5} \begin{pmatrix} 3/2 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 2 & 2 \\ 2 & -3 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} -8 & -3 \\ -8 & -7 \end{pmatrix}.
\]

b) \[
A^{100} = CD^{100}C^{-1} = \frac{1}{5} \begin{pmatrix} 3/2 & 1 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}^{100} \begin{pmatrix} 2 & 2 \\ 2 & -3 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 3 \cdot 2^{100} & 2 \cdot 2^{100} \\ 3 \cdot 2^{100} & 2^{101} - 3 \end{pmatrix}.
\]