1. a) Fill in: $A$ and $B$ are invertible $n \times n$ matrices, then the inverse of $AB$ is __________.

b) If the columns of an $n \times n$ matrix $Z$ are linearly independent, is $Z$ necessarily invertible? Justify your answer.

c) If $A$ and $B$ are $n \times n$ matrices and $ABx = 0$ has a unique solution, does $Ax = 0$ necessarily have a unique solution? Justify your answer.

Solution.

a) $(AB)^{-1} = B^{-1}A^{-1}$.

b) Yes. The transformation $x \rightarrow Zx$ is one-to-one since the columns of $Z$ are linearly independent. Thus $Z$ has a pivot in all $n$ columns, so $Z$ has $n$ pivots. Since $Z$ also has $n$ rows, this means that $Z$ has a pivot in every row, so $x \rightarrow Zx$ is onto. Therefore, $Z$ is invertible.

Alternatively, since $Z$ is an $n \times n$ matrix whose columns are linearly independent, the Invertible Matrix Theorem (2.3) in 2.3 says that $Z$ is invertible.

c) Yes. Since $AB$ is an $n \times n$ matrix and $ABx = 0$ has a unique solution, the Invertible Matrix Theorem says that $AB$ is invertible. Note $A$ is invertible and its inverse is $B(AB)^{-1}$, since these are square matrices and

$$A(B(AB)^{-1}) = AB(AB)^{-1} = I_n.$$ 

Since $A$ is invertible, $Ax = 0$ has a unique solution by the Invertible Matrix Theorem.

2. Let $A$ be an $n \times n$ matrix.

a) Using cofactor expansion, explain why $\det(A) = 0$ if $A$ has a row or a column of zeros.

b) Using cofactor expansion, explain why $\det(A) = 0$ if $A$ has adjacent identical columns.

Solution.

a) If $A$ has zeros for all entries in row $i$ (so $a_{i1} = a_{i2} = \cdots = a_{in} = 0$), then the cofactor expansion along row $i$ is

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} = 0 \cdot C_{i1} + 0 \cdot C_{i2} + \cdots + 0 \cdot C_{in} = 0.$$ 

Similarly, if $A$ has zeros for all entries in column $j$, then the cofactor expansion along column $j$ is the sum of a bunch of zeros and is thus 0.

b) If $A$ has identical adjacent columns, then the cofactor expansions will be identical except that one expansion’s terms for $\det(A)$ will have plus signs where
the other expansion’s terms for det(A) have minus signs (due to the \((-1)^{\text{power}}\) factors) and vice versa.

Therefore, \(\det(A) = -\det(A)\), so \(\det A = 0\).

3. Find the volume of the parallelepiped in \(\mathbb{R}^4\) naturally determined by the vectors

\[
\begin{bmatrix}
4 \\
1 \\
3 \\
8
\end{bmatrix},
\begin{bmatrix}
0 \\
7 \\
0 \\
3
\end{bmatrix},
\begin{bmatrix}
0 \\
2 \\
0 \\
1
\end{bmatrix},
\begin{bmatrix}
5 \\
2 \\
0 \\
7
\end{bmatrix}.
\]

Solution.

We put the vectors as columns of a matrix \(A\) and find \(\det(A)\). For this, we expand \(\det(A)\) along the third row because it only has one nonzero entry.

\[
\det(A) = 3(-1)^{3+1} \cdot \det \begin{bmatrix}
0 & 0 & 5 \\
7 & 2 & -5 \\
3 & 1 & 7
\end{bmatrix} = 3 \cdot 5(-1)^{1+3} \det \begin{bmatrix}
7 & 2 \\
3 & 1
\end{bmatrix} = 3(5)(1)(7-6) = 15.
\]

(In the second step, we used the cofactor expansion along the first row since it had only one nonzero entry.)

The volume is \(|\det(A)| = 15| = 15\).

4. If \(A\) is a \(3 \times 3\) matrix and \(\det(A) = 1\), what is \(\det(-2A)\)?

Solution.

By determinant properties, scaling one row by \(c\) multiplies the determinant by \(c\). When we take \(cA\) for an \(n \times n\) matrix \(A\), we are multiplying each row by \(c\). This multiplies the determinant by \(c\) a total of \(n\) times.

Thus, if \(A\) is \(n \times n\), then \(\det(cA) = c^n \det(A)\). Here \(n = 3\), so

\[
\det(-2A) = (-2)^3 \det(A) = -8 \det(A) = -8.
\]

5. a) Is there a real \(2 \times 2\) matrix \(A\) that satisfies \(A^4 = -I_2\)? Either write such an \(A\), or show that no such \(A\) exists.

(hint: think geometrically! The matrix \(-I_2\) represents rotation by \(\pi\) radians).

b) Is there a real \(3 \times 3\) matrix \(A\) that satisfies \(A^4 = -I_3\)? Either write such an \(A\), or show that no such \(A\) exists.

Solution.

a) Yes. Just take \(A\) to be the matrix of counterclockwise rotation by \(\pi/4\) radians:

\[
A = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.
\]
Then $A^2$ gives rotation c.c. by $\frac{\pi}{2}$ radians, $A^3$ gives rotation c.c. by $\frac{3\pi}{4}$ radians, and $A^4$ gives rotation c.c. by $\pi$ radians, which has matrix $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I_2$.

b) No. If $A^4 = -I$ then

$$[\det(A)]^4 = \det(A^4) = \det(-I) = (-1)^3 = -1.$$ 

In other words, if $A^4 = -I$ then $[\det(A)]^4 = -1$, which is impossible since $\det(A)$ is a real number.

Similarly, $A^4 = -I$ is impossible if $A$ is $5 \times 5$, $7 \times 7$, etc.