Math 1553 Supplement §3.5-3.7, 3.9, 4.1

Solutions

1. Justify why each of the following true statements can be checked without row reduction.

a) $\left\{ \begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \pi \end{pmatrix}, \begin{pmatrix} 0 \\ \sqrt{2} \\ 0 \end{pmatrix} \right\}$ is linearly independent.

b) $\left\{ \begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 10 \\ 20 \end{pmatrix}, \begin{pmatrix} 0 \\ 5 \\ 7 \end{pmatrix} \right\}$ is linearly independent.

c) $\left\{ \begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 10 \\ 20 \end{pmatrix}, \begin{pmatrix} 0 \\ 5 \\ 7 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ is linearly dependent.

Solution.

a) You can eyeball linear independence: if

$$x \begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix} + y \begin{pmatrix} 0 \\ 0 \\ \pi \end{pmatrix} + z \begin{pmatrix} 0 \\ \sqrt{2} \\ 0 \end{pmatrix} = \begin{pmatrix} 3x \\ 3x + y\sqrt{2} \\ 4x + \pi z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

then $x = 0$, so $y = z = 0$ too.

b) Since the first coordinate of $\begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix}$ is nonzero, $\begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix}$ cannot be in the span of $\left\{ \begin{pmatrix} 0 \\ 10 \\ 20 \end{pmatrix}, \begin{pmatrix} 0 \\ 5 \\ 7 \end{pmatrix} \right\}$. And $\begin{pmatrix} 0 \\ 10 \\ 20 \end{pmatrix}$ is not in the span of $\left\{ \begin{pmatrix} 0 \\ 5 \\ 7 \end{pmatrix} \right\}$ because it is not a multiple. Hence the span gets bigger each time you add a vector, so they're linearly independent.

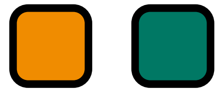
c) Any four vectors in $\mathbf{R}^3$ are linearly dependent; you don't need row reduction for that.

2. Consider the colors on the right. For which h is

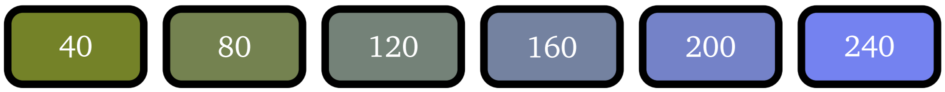
$$\left\{\begin{pmatrix}180\\50\\200\end{pmatrix},\begin{pmatrix}100\\150\\100\end{pmatrix},\begin{pmatrix}116\\130\\h\end{pmatrix}\right\}$$

linearly dependent? What does that say about the corresponding color?

$$\begin{pmatrix}180\\50\\200\end{pmatrix}\quad\begin{pmatrix}100\\150\\100\end{pmatrix}$$



$h=$



Solution.

The vectors

$$\begin{pmatrix}180\\50\\200\end{pmatrix},\quad\begin{pmatrix}100\\150\\100\end{pmatrix},\quad\begin{pmatrix}116\\130\\h\end{pmatrix}$$

are linearly dependent if and only if the vector equation

$$x\begin{pmatrix}180\\50\\200\end{pmatrix}+y\begin{pmatrix}100\\150\\100\end{pmatrix}+z\begin{pmatrix}116\\130\\h\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

has a nonzero solution. This translates into the matrix

$$\begin{pmatrix}180&100&116\\50&150&130\\200&100&h\end{pmatrix}\xrightarrow{\text{rref}}\begin{pmatrix}1&0&.2\\0&1&.8\\0&0&h-120\end{pmatrix},$$

which has a free variable if and only if $\boxed{h=120}$.

Suppose now that $h=120$. The parametric form for the solution the above vector equation is

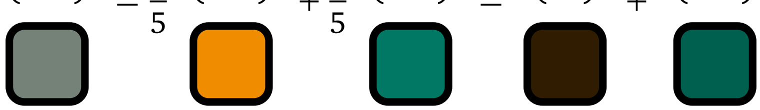
$$\begin{aligned}x&=-.2z\\y&=-.8z.\end{aligned}$$

Taking $z=1$ gives the linear combination

$$-.2\begin{pmatrix}180\\50\\200\end{pmatrix}-.8\begin{pmatrix}100\\150\\100\end{pmatrix}+\begin{pmatrix}116\\130\\120\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}.$$

In terms of colors:

$$\begin{pmatrix}116\\130\\120\end{pmatrix}=\frac{1}{5}\begin{pmatrix}180\\50\\200\end{pmatrix}+\frac{4}{5}\begin{pmatrix}100\\150\\100\end{pmatrix}=\begin{pmatrix}36\\10\\40\end{pmatrix}+\begin{pmatrix}80\\120\\80\end{pmatrix}$$



3. Find bases for the column space and the null space of

$$A = \begin{pmatrix} 0 & 1 & -3 & 1 & 0 \\ 1 & -1 & 8 & -7 & 1 \\ -1 & -2 & 1 & 4 & -1 \end{pmatrix}.$$

Solution.

The RREF of $\left(A \mid 0\right)$ is

$$\left(\begin{array}{ccccc|c} 1 & 0 & 5 & -6 & 1 & 0 \\ 0 & 1 & -3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right),$$

so x_3, x_4, x_5 are free, and

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -5x_3 + 6x_4 - x_5 \\ 3x_3 - x_4 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = x_3 \begin{pmatrix} -5 \\ 3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 6 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Therefore, a basis for Nul A is $\left\{ \begin{pmatrix} -5 \\ 3 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 6 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$.

To find a basis for Col A, we use the pivot columns as they were written in the *original* matrix A, not its RREF. These are the first two columns:

$$\left\{ \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \right\}.$$

4. Find a basis for the subspace V of $\mathbf{R}^4$ given by

$$V = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \text{ in } \mathbf{R}^4 \;\middle|\; x + 2y - 3z + w = 0 \right\}.$$

Solution.

V is Nul A for the 1×4 matrix $A = \begin{pmatrix} 1 & 2 & -3 & 1 \end{pmatrix}$. The augmented matrix $\left(A \mid 0\right) = \left(\begin{array}{cccc|c} 1 & 2 & -3 & 1 & 0 \end{array}\right)$ gives $x = -2y + 3z - w$ where y, z, w are free variables. The parametric vector form for the solution set to $Ax = 0$ is

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -2y + 3z - w \\ y \\ z \\ w \end{pmatrix} = y \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + z \begin{pmatrix} 3 \\ 0 \\ 1 \\ 0 \end{pmatrix} + w \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Therefore, a basis for V is

$$\left\{\begin{pmatrix}-2\\1\\0\\0\end{pmatrix},\begin{pmatrix}3\\0\\1\\0\end{pmatrix},\begin{pmatrix}-1\\0\\0\\1\end{pmatrix}\right\}.$$

5. **a)** True or false: If A is an $m\times n$ matrix and $\mathrm{Nul}(A)=\mathbf{R}^n$, then $\mathrm{Col}(A)=\{0\}$.

b) Give an example of 2×2 matrix whose column space is the same as its null space.

Solution.

a) If $\mathrm{Nul}(A)=\mathbf{R}^n$ then $Ax=0$ for all x in $\mathbf{R}^n$, so the only element in $\mathrm{Col}(A)$ is $\{0\}$. Alternatively, the rank theorem says

$$\dim(\mathrm{Col}\ A)+\dim(\mathrm{Nul}\ A)=n \implies \dim(\mathrm{Col}\ A)+n=n \implies \dim(\mathrm{Col}\ A)=0 \implies \mathrm{Col}\ A=\{0\}.$$

b) Take $A=\begin{pmatrix}0&1\\0&0\end{pmatrix}$. Its null space and column space are $\mathrm{Span}\left\{\begin{pmatrix}1\\0\end{pmatrix}\right\}$.

6. For each matrix A, describe what the transformation $T(x)=Ax$ does to $\mathbf{R}^3$ geometrically.

$$\textbf{a)}\begin{pmatrix}-1&0&0\\0&1&0\\0&0&1\end{pmatrix}\qquad \textbf{b)}\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}.$$

Solution.

a) We compute

$$T\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}-1&0&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}-x\\y\\z\end{pmatrix}.$$

This is the reflection over the yz-plane.

b) We compute

$$T\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\z\end{pmatrix}.$$

This is projection onto the z-axis.