The third midterm is on this **Friday, November 16**.
- The exam covers §§4.5, 5.1, 5.2, 5.3, 6.1, 6.2, 6.4, 6.5.
- About half the problems will be conceptual, and the other half computational.

**WeBWorK 6.4, 6.5 are due today at 11:59pm.**

There is a practice midterm posted on the website. It is meant to be similar in format and difficulty to the real midterm.

**Study tips:**
- Drill problems in Lay. Practice the recipes until you can do them in your sleep.
- Make sure to learn the theorems and learn the definitions, and understand what they mean. Make flashcards!
- There's a list of items to review at the beginning of every section of the book.
- Sit down to do the practice midterm in 50 minutes, with no notes.
- Come to office hours!

**TA review session:** Skiles 202, Thursday, 7–8pm.

**My office is Skiles 244 and Rabinoffice hours are:** Mondays, 12–1pm; Wednesdays, 1–3pm. **Extra office hours:** Thursday, 9–11am.
Chapter 7

Orthogonality
Section 7.1

Dot Products and Orthogonality
Recall: This course is about learning to:

- Solve the matrix equation $Ax = b$
- Solve the matrix equation $Ax = \lambda x$
- Almost solve the equation $Ax = b$

We are now aiming at the last topic.

Idea: In the real world, data is imperfect. Suppose you measure a data point $x$ which you know for theoretical reasons must lie on a plane spanned by two vectors $u$ and $v$.

Due to measurement error, though, the measured $x$ is not actually in $\text{Span}\{u, v\}$. In other words, the equation $au + bv = x$ has no solution. What do you do? The real value is probably the closest point to $x$ on $\text{Span}\{u, v\}$. Which point is that?
The Dot Product

We need a notion of *angle* between two vectors, and in particular, a notion of *orthogonality* (i.e. when two vectors are perpendicular). This is the purpose of the dot product.

**Definition**

The **dot product** of two vectors $x, y$ in $\mathbb{R}^n$ is

$$ x \cdot y = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \overset{\text{def}}{=} x_1 y_1 + x_2 y_2 + \cdots + x_n y_n. $$

Thinking of $x, y$ as column vectors, this is the same as $x^T y$.

**Example**

$$ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} = 1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6 = 32. $$
Properties of the Dot Product

Many usual arithmetic rules hold, as long as you remember you can only dot two vectors together, and that the result is a scalar.

- $x \cdot y = y \cdot x$
- $(x + y) \cdot z = x \cdot z + y \cdot z$
- $(cx) \cdot y = c(x \cdot y)$

Dotting a vector with itself is special:

$$
\begin{pmatrix}
  x_1 \\
  x_2 \\
  \vdots \\
  x_n
\end{pmatrix}
\cdot
\begin{pmatrix}
  x_1 \\
  x_2 \\
  \vdots \\
  x_n
\end{pmatrix}
= x_1^2 + x_2^2 + \cdots + x_n^2.
$$

Hence:

- $x \cdot x \geq 0$
- $x \cdot x = 0$ if and only if $x = 0$.

Important: $x \cdot y = 0$ does not imply $x = 0$ or $y = 0$. For example, $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$. 
The Dot Product and Length

Definition

The **length** or **norm** of a vector $x$ in $\mathbb{R}^n$ is

$$
\|x\| = \sqrt{x \cdot x} = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}.
$$

Why is this a good definition? The Pythagorean theorem!

Fact

If $x$ is a vector and $c$ is a scalar, then $\|cx\| = |c| \cdot \|x\|.$

$$
\left\| \begin{pmatrix} 6 \\ 8 \end{pmatrix} \right\| = 2 \left\| \begin{pmatrix} 3 \\ 4 \end{pmatrix} \right\| = 2 \left\| \begin{pmatrix} 3 \\ 4 \end{pmatrix} \right\| = 10
$$
The Dot Product and Distance

Definition
The **distance** between two points $x, y$ in $\mathbb{R}^n$ is

$$\text{dist}(x, y) = ||y - x||.$$  

This is just the length of the vector from $x$ to $y$.

Example
Let $x = (1, 2)$ and $y = (4, 4)$. Then

$$\text{dist}(x, y) = ||y - x|| = \left\| \begin{pmatrix} 3 \\ 2 \end{pmatrix} \right\| = \sqrt{3^2 + 2^2} = \sqrt{13}.$$  

![Diagram showing the distance between points x and y]
Unit Vectors

Definition
A **unit vector** is a vector \( \mathbf{v} \) with length \( \| \mathbf{v} \| = 1 \).

Example
The unit coordinate vectors are unit vectors:

\[
\| \mathbf{e}_1 \| = \left\| \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\| = \sqrt{1^2 + 0^2 + 0^2} = 1
\]

Definition
Let \( \mathbf{x} \) be a nonzero vector in \( \mathbb{R}^n \). The **unit vector in the direction of** \( \mathbf{x} \) is the vector \( \frac{\mathbf{x}}{\| \mathbf{x} \|} \).

This is in fact a unit vector:

\[
\left\| \frac{\mathbf{x}}{\| \mathbf{x} \|} \right\| = \frac{1}{\| \mathbf{x} \|} \| \mathbf{x} \| = 1.
\]
Example

What is the unit vector in the direction of \( \mathbf{x} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \)?

\[
\mathbf{u} = \frac{\mathbf{x}}{\|\mathbf{x}\|} = \frac{1}{\sqrt{3^2 + 4^2}} \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 3 \\ 4 \end{pmatrix}.
\]
Orthogonality

**Definition**
Two vectors $x, y$ are **orthogonal** or **perpendicular** if $x \cdot y = 0$.

*Notation:* $x \perp y$ means $x \cdot y = 0$.

Why is this a good definition? The Pythagorean theorem / law of cosines!

**Law of cosines:**
$$\|x - y\|^2 = \|x\|^2 + \|y\|^2 - 2\|x\|\|y\|\cos \alpha$$

$\alpha = 90^\circ \iff \cos \alpha = 0$

$x$ and $y$ are perpendicular $\iff \|x\|^2 + \|y\|^2 = \|x - y\|^2$

$\iff x \cdot x + y \cdot y = (x - y) \cdot (x - y)$

$\iff x \cdot x + y \cdot y = x \cdot x + y \cdot y - 2x \cdot y$

$\iff x \cdot y = 0$

**Fact:** $x \perp y \iff \|x - y\|^2 = \|x\|^2 + \|y\|^2$
Problem: Find all vectors orthogonal to \( \mathbf{v} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \).

We have to find all vectors \( \mathbf{x} \) such that \( \mathbf{x} \cdot \mathbf{v} = 0 \). This means solving the equation

\[
0 = \mathbf{x} \cdot \mathbf{v} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = x_1 + x_2 - x_3.
\]

The parametric form for the solution is \( x_1 = -x_2 + x_3 \), so the parametric vector form of the general solution is

\[
\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.
\]

For instance, \( \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \perp \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \) because \( \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = 0 \).
Problem: Find all vectors orthogonal to both \( v = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \) and \( w = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \).

Now we have to solve the system of two homogeneous equations

\[
0 = x \cdot v = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = x_1 + x_2 - x_3
\]

\[
0 = x \cdot w = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = x_1 + x_2 + x_3.
\]

In matrix form:

\[
\begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\]

The rows are \( v \) and \( w \)

The parametric vector form of the solution is

\[
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}.
\]
Orthogonality

General procedure

**Problem:** Find all vectors orthogonal to some number of vectors $v_1, v_2, \ldots, v_m$ in $\mathbb{R}^n$.

This is the same as finding all vectors $x$ such that

$$0 = v_1^T x = v_2^T x = \cdots = v_m^T x.$$

Putting the *row* vectors $v_1^T, v_2^T, \ldots, v_m^T$ into a matrix, this is the same as finding all $x$ such that

$$\begin{pmatrix}
- v_1^T \\
- v_2^T \\
\vdots \\
- v_m^T
\end{pmatrix} x = \begin{pmatrix}
v_1 \cdot x \\
v_2 \cdot x \\
\vdots \\
v_m \cdot x
\end{pmatrix} = 0.$$

The set of all vectors orthogonal to some vectors $v_1, v_2, \ldots, v_m$ in $\mathbb{R}^n$ is the *null space* of the $m \times n$ matrix you get by “turning them sideways and smooshing them together:”

In particular, this set is a subspace!
The **dot product** of vectors $x, y$ in $\mathbb{R}^n$ is the number $x^T y$.

The **length** or **norm** of a vector $x$ in $\mathbb{R}^n$ is $\|x\| = \sqrt{x \cdot x}$.

The **distance** between two vectors $x, y$ in $\mathbb{R}^n$ is $\text{dist}(x, y) = \|y - x\|$.

A **unit vector** is a vector $v$ with length $\|v\| = 1$.

The **unit vector in the direction of** $x$ is $x/\|x\|$.

Two vectors $x, y$ are **orthogonal** if $x \cdot y = 0$.

The set of all vectors orthogonal to some vectors $v_1, v_2, \ldots, v_m$ in $\mathbb{R}^n$ is the null space of the matrix

$$
\begin{pmatrix}
- v_1^T \\
- v_2^T \\
\vdots \\
- v_m^T
\end{pmatrix}.
$$