Announcements
Monday, November 5

▶ The third midterm is on **Friday, November 16**.
  ▶ That is one week from this Friday.
  ▶ The exam covers §§4.5, 5.1, 5.2, 5.3, 6.1, 6.2, 6.4, 6.5.

▶ WeBWorK 6.1, 6.2 are due Wednesday at 11:59pm.

▶ The quiz on Friday covers §§6.1, 6.2.

▶ My office is Skiles 244 and Rabinoffice hours are: Mondays, 12–1pm; Wednesdays, 1–3pm.
Section 6.4

Diagonalization
Many real-word linear algebra problems have the form:

\[ v_1 = A v_0, \quad v_2 = Av_1 = A^2 v_0, \quad v_3 = Av_2 = A^3 v_0, \quad \ldots \quad v_n = Av_{n-1} = A^n v_0. \]

This is called a **difference equation**.

Our toy example about rabbit populations had this form.

The question is, what happens to \( v_n \) as \( n \to \infty \)?

- Taking powers of diagonal matrices is easy!
- Taking powers of *diagonalizable* matrices is still easy!
- Diagonalizing a matrix is an eigenvalue problem.
If $D$ is diagonal, then $D^n$ is also diagonal; its diagonal entries are the $n$th powers of the diagonal entries of $D$: 

\[
D = \begin{pmatrix}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}, \\
D^2 = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}, \\
D^3 = \begin{pmatrix}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}, \\
\ldots \\
D^n = \begin{pmatrix}
(-1)^n & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]
Powers of Matrices that are Similar to Diagonal Ones

What if $A$ is not diagonal?

Example
Let $A = \begin{pmatrix} 1/2 & 3/2 \\ 3/2 & 1/2 \end{pmatrix}$. Compute $A^n$, using

$$A = CDC^{-1}$$

for $C = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ and $D = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$.

We compute:

$$A^2 =$$

$$A^3 =$$

$$\vdots$$

$$A^n =$$

Therefore

$$A^n =$$
Similar Matrices

Definition
Two \( n \times n \) matrices are similar if there exists an invertible \( n \times n \) matrix \( C \) such that \( A = CBC^{-1} \).

Fact: if two matrices are similar then so are their powers:

\[
A = CBC^{-1} \implies A^n = CB^n C^{-1}.
\]

Fact: if \( A \) is similar to \( B \) and \( B \) is similar to \( D \), then \( A \) is similar to \( D \).
Definition

An \( n \times n \) matrix \( A \) is **diagonalizable** if it is similar to a diagonal matrix:

\[
A = CDC^{-1} \quad \text{for } D \text{ diagonal.}
\]

**Important**

If \( A = CDC^{-1} \) for \( D = \begin{pmatrix}
    d_{11} & 0 & \cdots & 0 \\
    0 & d_{22} & \cdots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \cdots & d_{nn}
\end{pmatrix} \) then

\[
A^k = CD^kC^{-1} = C \begin{pmatrix}
    d_{11}^k & 0 & \cdots & 0 \\
    0 & d_{22}^k & \cdots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \cdots & d_{nn}^k
\end{pmatrix} C^{-1}.
\]

So diagonalizable matrices are easy to raise to any power.
The Diagonalization Theorem

An $n \times n$ matrix $A$ is diagonalizable if and only if $A$ has $n$ linearly independent eigenvectors.

In this case, $A = CDC^{-1}$ for

$$C = \begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix} \quad D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix},$$

where $v_1, v_2, \ldots, v_n$ are linearly independent eigenvectors, and $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the corresponding eigenvalues (in the same order).

Corollary

A theorem that follows easily from another theorem

An $n \times n$ matrix with $n$ distinct eigenvalues is diagonalizable.

The Corollary is true because eigenvectors with distinct eigenvalues are always linearly independent. We will see later that a diagonalizable matrix need not have $n$ distinct eigenvalues though.
The Diagonalization Theorem

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In this case, $A = CDC^{-1}$ for

$$C = \begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix} \quad D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix},$$

where $v_1, v_2, \ldots, v_n$ are linearly independent eigenvectors, and $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the corresponding eigenvalues (in the same order).

Note that the decomposition is not unique: you can reorder the eigenvalues and eigenvectors.

$$A = \begin{pmatrix} v_1 & v_2 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} v_1 & v_2 \end{pmatrix}^{-1} = \begin{pmatrix} v_2 & v_1 \end{pmatrix} \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_1 \end{pmatrix} \begin{pmatrix} v_2 & v_1 \end{pmatrix}^{-1}$$
Diagonalization

Easy example

**Question:** What does the Diagonalization Theorem say about the matrix

\[ A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \]?

This is a triangular matrix, so the eigenvalues are the diagonal entries 1, 2, 3.

A diagonal matrix just scales the coordinates by the diagonal entries, so we can take our eigenvectors to be the unit coordinate vectors \( e_1, e_2, e_3 \). Hence the Diagonalization Theorem says

\[ A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \]

It doesn't give us anything new because the matrix was already diagonal!

A diagonal matrix \( D \) is diagonalizable! It is similar to itself:

\[ D = I_n D I_n^{-1}. \]
Diagonalization
Example

**Problem:** Diagonalize \( A = \begin{pmatrix} 1/2 & 3/2 \\ 3/2 & 1/2 \end{pmatrix} \).
Diagonalization

Another example

Problem: Diagonalize $A = \begin{pmatrix} 4 & -3 & 0 \\ 2 & -1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$. 

The characteristic polynomial is $f(\lambda) = \det(A - \lambda I) = -\lambda^3 + 4\lambda^2 - 5\lambda + 2 = -\lambda(\lambda - 1)^2(\lambda - 2)$. Therefore the eigenvalues are 1 and 2, with respective multiplicities 2 and 1.

Let's compute the 1-eigenspace: $(A - I)x = 0 \iff \begin{pmatrix} 3 & -3 & 0 \\ 2 & -2 & 0 \\ 1 & -1 & 0 \end{pmatrix}x = 0$. 

$rref \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}x = 0$ 

The parametric vector form is $x = y, y = z \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = y\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. 

Hence a basis for the 1-eigenspace is $B_1 = \{ v_1, v_2 \}$ where $v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. 
Problem: Diagonalize $A = \begin{pmatrix} 4 & -3 & 0 \\ 2 & -1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$.

Note: In this case, there are three linearly independent eigenvectors, but only two distinct eigenvalues.
Problem: Show that $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is not diagonalizable.

Conclusion: $A$ has only one linearly independent eigenvector, so by the “only if” part of the diagonalization theorem, $A$ is not diagonalizable.
Which of the following matrices are diagonalizable, and why?

A. \[
\begin{pmatrix}
1 & 2 \\
0 & 1
\end{pmatrix}
\]

B. \[
\begin{pmatrix}
1 & 2 \\
0 & 2
\end{pmatrix}
\]

C. \[
\begin{pmatrix}
2 & 1 \\
0 & 2
\end{pmatrix}
\]

D. \[
\begin{pmatrix}
2 & 0 \\
0 & 2
\end{pmatrix}
\]

Matrix A is not diagonalizable: its only eigenvalue is 1, and its 1-eigenspace is spanned by \[
\begin{pmatrix}
1 \\
0
\end{pmatrix}
\].

Similarly, matrix C is not diagonalizable.

Matrix B is diagonalizable because it is a 2 \times 2 matrix with distinct eigenvalues.

Matrix D is already diagonal!
How to diagonalize a matrix $A$:

1. Find the eigenvalues of $A$ using the characteristic polynomial.
2. For each eigenvalue $\lambda$ of $A$, compute a basis $B_\lambda$ for the $\lambda$-eigenspace.
3. If there are fewer than $n$ total vectors in the union of all of the eigenspace bases $B_\lambda$, then the matrix is not diagonalizable.
4. Otherwise, the $n$ vectors $v_1, v_2, \ldots, v_n$ in your eigenspace bases are linearly independent, and $A = CDC^{-1}$ for

$$C = \begin{pmatrix} | & | & \cdots & | \hline v_1 & v_2 & \cdots & v_n \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix},$$

where $\lambda_i$ is the eigenvalue for $v_i$. 

Why is the Diagonalization Theorem true?

Proof

A diagonalizable implies $A$ has $n$ linearly independent eigenvectors: Suppose $A = CD^{-1}$, where $D$ is diagonal with diagonal entries $\lambda_1, \lambda_2, \ldots, \lambda_n$. Let $v_1, v_2, \ldots, v_n$ be the columns of $C$. They are linearly independent because $C$ is invertible. So $Ce_i = v_i$, hence $C^{-1}v_i = e_i$.

$Av_i = CDC^{-1}v_i = CDe_i = C(\lambda_i e_i) = \lambda_i Ce_i = \lambda_i v_i$.

Hence $v_i$ is an eigenvector of $A$ with eigenvalue $\lambda_i$. So the columns of $C$ form $n$ linearly independent eigenvectors of $A$, and the diagonal entries of $D$ are the eigenvalues.

A has $n$ linearly independent eigenvectors implies $A$ is diagonalizable: Suppose $A$ has $n$ linearly independent eigenvectors $v_1, v_2, \ldots, v_n$, with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Let $C$ be the invertible matrix with columns $v_1, v_2, \ldots, v_n$. Let $D = C^{-1}AC$.

$De_i = C^{-1}ACe_i = C^{-1}Av_i = C^{-1}(\lambda_i v_i) = \lambda_i C^{-1}v_i = \lambda_i e_i$.

Hence $D$ is diagonal, with diagonal entries $\lambda_1, \lambda_2, \ldots, \lambda_n$. Solving $D = C^{-1}AC$ for $A$ gives $A = CDC^{-1}$. 
Non-Distinct Eigenvalues

Definition
Let \( \lambda \) be an eigenvalue of a square matrix \( A \). The **geometric multiplicity** of \( \lambda \) is the dimension of the \( \lambda \)-eigenspace.

Theorem
Let \( \lambda \) be an eigenvalue of a square matrix \( A \). Then

\[
1 \leq \text{(the geometric multiplicity of } \lambda) \leq \text{(the algebraic multiplicity of } \lambda).
\]

The proof is beyond the scope of this course.

Corollary
Let \( \lambda \) be an eigenvalue of a square matrix \( A \). If the algebraic multiplicity of \( \lambda \) is 1, then the geometric multiplicity is also 1: the eigenspace is a line.

The Diagonalization Theorem (Alternate Form)
Let \( A \) be an \( n \times n \) matrix. The following are equivalent:

1. \( A \) is diagonalizable.
2. The sum of the geometric multiplicities of the eigenvalues of \( A \) equals \( n \).
3. The sum of the algebraic multiplicities of the eigenvalues of \( A \) equals \( n \), and the geometric multiplicity equals the algebraic multiplicity of each eigenvalue.
Non-Distinct Eigenvalues

Examples

Example

If $A$ has $n$ distinct eigenvalues, then the algebraic multiplicity of each equals 1, hence so does the geometric multiplicity, and therefore $A$ is diagonalizable.

For example, $A = \begin{pmatrix} 1/2 & 3/2 \\ 3/2 & 1/2 \end{pmatrix}$ has eigenvalues $-1$ and $2$, so it is diagonalizable.

Example

The matrix $A = \begin{pmatrix} 4 & -3 & 0 \\ 2 & -1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$ has characteristic polynomial

$$f(\lambda) = - (\lambda - 1)^2(\lambda - 2).$$

The algebraic multiplicities of 1 and 2 are 2 and 1, respectively. They sum to 3. We showed before that the geometric multiplicity of 1 is 2 (the 1-eigenspace has dimension 2). The eigenvalue 2 automatically has geometric multiplicity 1. Hence the geometric multiplicities add up to 3, so $A$ is diagonalizable.
Example

The matrix \( A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \) has characteristic polynomial \( f(\lambda) = (\lambda - 1)^2 \).

It has one eigenvalue 1 of algebraic multiplicity 2.

We showed before that the geometric multiplicity of 1 is 1 (the 1-eigenspace has dimension 1).

Since the geometric multiplicity is smaller than the algebraic multiplicity, the matrix is not diagonalizable.
A matrix $A$ is **diagonalizable** if it is similar to a diagonal matrix $D$: $A = CDC^{-1}$.

It is easy to take powers of diagonalizable matrices: $A^r = CD^r C^{-1}$.

An $n \times n$ matrix is diagonalizable if and only if it has $n$ linearly independent eigenvectors $v_1, v_2, \ldots, v_n$, in which case $A = CDC^{-1}$ for

$$
C = \begin{pmatrix}
v_1 & v_2 & \cdots & v_n \\
\end{pmatrix}, \\
D = \begin{pmatrix}
\lambda_1 & 0 & \cdots & 0 \\
0 & \lambda_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \lambda_n
\end{pmatrix}.
$$

If $A$ has $n$ distinct eigenvalues, then it is diagonalizable.

The **geometric multiplicity** of an eigenvalue $\lambda$ is the dimension of the $\lambda$-eigenspace.

$1 \leq \text{(geometric multiplicity)} \leq \text{(algebraic multiplicity)}$.

An $n \times n$ matrix is diagonalizable if and only if the sum of the geometric multiplicities is $n$. 