WeBWorK on determinants due today at 11:59pm.

The quiz on Friday covers §§5.1, 5.2, 5.3.

My office is Skiles 244 and Rabin office hours are: Mondays, 12–1pm; Wednesdays, 1–3pm.
**Definition**

Let $A$ be an $n \times n$ matrix.

1. An **eigenvector** of $A$ is a nonzero vector $v$ in $\mathbb{R}^n$ such that $Av = \lambda v$, for some $\lambda$ in $\mathbb{R}$.

2. An **eigenvalue** of $A$ is a number $\lambda$ in $\mathbb{R}$ such that the equation $Av = \lambda v$ has a nontrivial solution.

3. If $\lambda$ is an eigenvalue of $A$, the **$\lambda$-eigenspace** is the solution set of $(A - \lambda I_n)x = 0$. 
An eigenvector of a matrix $A$ is a nonzero vector $v$ such that:

- $Av$ is a multiple of $v$, which means $Av$ is collinear with $v$, which means $Av$ and $v$ are on the same line through the origin.

$v$ is an eigenvector

$w$ is not an eigenvector
Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be reflection over the line $L$ defined by $y = -x$, and let $A$ be the matrix for $T$.

**Question:** What are the eigenvalues and eigenspaces of $A$? No computations!

Does anyone see any eigenvectors (vectors that don’t move off their line)? $v$ is an eigenvector with eigenvalue $-1$. 

[vw]
Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be reflection over the line $L$ defined by $y = -x$, and let $A$ be the matrix for $T$.

**Question:** What are the eigenvalues and eigenspaces of $A$? No computations!

Does anyone see any eigenvectors (vectors that don’t move off their line)?

$w$ is an eigenvector with eigenvalue 1.
Eigenspaces
Geometry; example

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be reflection over the line $L$ defined by $y = -x$, and let $A$ be
the matrix for $T$.

Question: What are the eigenvalues and eigenspaces of $A$? No computations!

Does anyone see any eigenvectors (vectors that don’t move off their line)?

$u$ is not an eigenvector.
Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be reflection over the line $L$ defined by $y = -x$, and let $A$ be the matrix for $T$.

**Question:** What are the eigenvalues and eigenspaces of $A$? No computations!

Does anyone see any eigenvectors (vectors that don’t move off their line)?

Neither is $z$. 

[interactive]
Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be reflection over the line $L$ defined by $y = -x$, and let $A$ be the matrix for $T$.

**Question:** What are the eigenvalues and eigenspaces of $A$? No computations!

Does anyone see any eigenvectors (vectors that don’t move off their line)?

The 1-eigenspace is $L$
(all the vectors $x$ where $Ax = x$).
Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be reflection over the line $L$ defined by $y = -x$, and let $A$ be the matrix for $T$.

**Question:** What are the eigenvalues and eigenspaces of $A$? No computations!

Does anyone see any eigenvectors (vectors that don’t move off their line)?

The $(-1)$-eigenspace is the line $y = x$ (all the vectors $x$ where $Ax = -x$).
Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the vertical projection onto the $x$-axis, and let $A$ be the matrix for $T$.

**Question:** What are the eigenvalues and eigenspaces of $A$? No computations!

Does anyone see any eigenvectors (vectors that don’t move off their line)?

$v$ is an eigenvector with eigenvalue 0.
Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be the vertical projection onto the $x$-axis, and let $A$ be the matrix for $T$.

**Question:** What are the eigenvalues and eigenspaces of $A$? No computations!

Does anyone see any eigenvectors (vectors that don’t move off their line)?

$w$ is an eigenvector with eigenvalue 1.
Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be the vertical projection onto the $x$-axis, and let $A$ be the matrix for $T$.

**Question:** What are the eigenvalues and eigenspaces of $A$? No computations!

Does anyone see any eigenvectors (vectors that don’t move off their line)?

$u$ is *not* an eigenvector.
Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be the vertical projection onto the $x$-axis, and let $A$ be the matrix for $T$.

**Question:** What are the eigenvalues and eigenspaces of $A$? No computations!

Does anyone see any eigenvectors (vectors that don’t move off their line)?

Neither is $z$. 

([interactive])
Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the vertical projection onto the $x$-axis, and let $A$ be the matrix for $T$.

**Question**: What are the eigenvalues and eigenspaces of $A$? No computations!

Does anyone see any eigenvectors (vectors that don’t move off their line)?

The 1-eigenspace is the $x$-axis (all the vectors $x$ where $Ax = x$).
Eigenspaces
Geometry; example

Let \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) be the vertical projection onto the \( x \)-axis, and let \( A \) be the matrix for \( T \).

**Question:** What are the eigenvalues and eigenspaces of \( A \)? No computations!

Does anyone see any eigenvectors (vectors that don’t move off their line)?

The 0-eigenspace is the **y-axis**
(all the vectors \( x \) where \( Ax = 0x \)).
Let
\[ A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \]
so \( T(x) = Ax \) is a shear in the \( x \)-direction.

**Question:** What are the eigenvalues and eigenspaces of \( A \)? No computations!

Does anyone see any eigenvectors (vectors that don’t move off their line)?

Vectors \( v \) above the \( x \)-axis are moved right but not up…

so they’re not eigenvectors.

[interactive]
Let

\[ A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \]

so \( T(x) = Ax \) is a shear in the \( x \)-direction.

**Question:** What are the eigenvalues and eigenspaces of \( A \)? No computations!

Does anyone see any eigenvectors (vectors that don’t move off their line)?

Vectors \( w \) below the \( x \)-axis are moved left but not down... so they’re not eigenvectors.
Let

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

so $T(x) = Ax$ is a shear in the $x$-direction.

**Question:** What are the eigenvalues and eigenspaces of $A$? No computations!

Does anyone see any eigenvectors (vectors that don’t move off their line)?

$u$ is an eigenvector with eigenvalue 1.
Let

\[ A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \]

so \( T(x) = Ax \) is a shear in the x-direction.

**Question:** What are the eigenvalues and eigenspaces of \( A \)? No computations!

Does anyone see any eigenvectors (vectors that don’t move off their line)?

The 1-eigenspace is the x-axis (all the vectors \( x \) where \( Ax = x \)).
Eigenspaces
Geometry; example

Let

\[ A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \]

so \( T(x) = Ax \) is a shear in the \( x \)-direction.

Question: What are the eigenvalues and eigenspaces of \( A \)? No computations!

Does anyone see any eigenvectors (vectors that don’t move off their line)?

There are no other eigenvectors.
Let \( T : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) be counterclockwise rotation by 45°, and let \( A \) be the matrix for \( T \).

Find an eigenvector of \( A \) without doing any computations.

A. Okay.  
B. No way.

Answer: B. No way. There are no eigenvectors!
Section 6.2

The Characteristic Polynomial
The Characteristic Polynomial

Let $A$ be a square matrix.

\[ \lambda \text{ is an eigenvalue of } A \iff Ax = \lambda x \text{ has a nontrivial solution} \]
\[ \iff (A - \lambda I)x = 0 \text{ has a nontrivial solution} \]
\[ \iff A - \lambda I \text{ is not invertible} \]
\[ \iff \det(A - \lambda I) = 0. \]

This gives us a way to compute the eigenvalues of $A$.

Definition
Let $A$ be a square matrix. The characteristic polynomial of $A$ is

\[ f(\lambda) = \det(A - \lambda I). \]

The characteristic equation of $A$ is the equation

\[ f(\lambda) = \det(A - \lambda I) = 0. \]

Important
The eigenvalues of $A$ are the roots of the characteristic polynomial $f(\lambda) = \det(A - \lambda I)$. 
The Characteristic Polynomial

Example

**Question:** What are the eigenvalues of

\[ A = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} \]?

**Answer:** First we find the characteristic polynomial:

\[
f(\lambda) = \det(A - \lambda I) = \det \left[ \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right] = \det \begin{pmatrix} 5 - \lambda & 2 \\ 2 & 1 - \lambda \end{pmatrix}
\]

\[
= (5 - \lambda)(1 - \lambda) - 2 \cdot 2 \\
= \lambda^2 - 6\lambda + 1.
\]

The eigenvalues are the roots of the characteristic polynomial, which we can find using the quadratic formula:

\[
\lambda = \frac{6 \pm \sqrt{36 - 4}}{2} = 3 \pm 2\sqrt{2}.
\]
Question: What is the characteristic polynomial of 

\[ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \]?

Answer:

\[ f(\lambda) = \det(A - \lambda I) = \det \begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix} = (a - \lambda)(d - \lambda) - bc \]

\[ = \lambda^2 - (a + d)\lambda + (ad - bc) \]

What do you notice about \( f(\lambda) \)?

- The constant term is \( \det(A) \), which is zero if and only if \( \lambda = 0 \) is a root.
- The linear term \( -(a + d) \) is the negative of the sum of the diagonal entries of \( A \).

Definition

The **trace** of a square matrix \( A \) is \( \text{Tr}(A) = \text{sum of the diagonal entries of } A \).

Shortcut

The characteristic polynomial of a 2×2 matrix \( A \) is

\[ f(\lambda) = \lambda^2 - \text{Tr}(A)\lambda + \det(A). \]
The Characteristic Polynomial

Example

**Question:** What are the eigenvalues of the rabbit population matrix 

\[ A = \begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \]?

**Answer:** First we find the characteristic polynomial:

\[ f(\lambda) = \det(A - \lambda I) = \det \begin{pmatrix} -\lambda & 6 & 8 \\ \frac{1}{2} & -\lambda & 0 \\ 0 & \frac{1}{2} & -\lambda \end{pmatrix} \]

\[ = 8 \left( \frac{1}{4} - 0 \cdot -\lambda \right) - \lambda \left( \lambda^2 - 6 \cdot \frac{1}{2} \right) \]

\[ = -\lambda^3 + 3\lambda + 2. \]

We know from before that one eigenvalue is \( \lambda = 2 \): indeed, \( f(2) = -8 + 6 + 2 = 0 \). Doing polynomial long division, we get:

\[ \frac{-\lambda^3 + 3\lambda + 2}{\lambda - 2} = -\lambda^2 - 2\lambda - 1 = -(\lambda + 1)^2. \]

Hence \( \lambda = -1 \) is also an eigenvalue.
**Definition**
The (algebraic) multiplicity of an eigenvalue $\lambda$ is its multiplicity as a root of the characteristic polynomial.

This is not a very interesting notion yet. It will become interesting when we also define geometric multiplicity later.

**Example**
In the rabbit population matrix, $f(\lambda) = -(\lambda - 2)(\lambda + 1)^2$, so the algebraic multiplicity of the eigenvalue 2 is 1, and the algebraic multiplicity of the eigenvalue $-1$ is 2.

**Example**
In the matrix $\begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$, $f(\lambda) = (\lambda - (3 - 2\sqrt{2}))(\lambda - (3 + 2\sqrt{2}))$, so the algebraic multiplicity of $3 + 2\sqrt{2}$ is 1, and the algebraic multiplicity of $3 - 2\sqrt{2}$ is 1.
Fact: If $A$ is an $n \times n$ matrix, the characteristic polynomial

$$f(\lambda) = \det(A - \lambda I)$$

turns out to be a polynomial of degree $n$, and its roots are the eigenvalues of $A$:

$$f(\lambda) = (-1)^n \lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + \cdots + a_1\lambda + a_0.$$
It’s easy to factor quadratic polynomials:

\[ x^2 + bx + c = 0 \implies x = \frac{-b \pm \sqrt{b^2 - 4c}}{2}. \]

It’s less easy to factor cubics, quartics, and so on:

\[ x^3 + bx^2 + cx + d = 0 \implies x = \text{???} \]
\[ x^4 + bx^3 + cx^2 + dx + e = 0 \implies x = \text{???} \]

Read about factoring polynomials by hand in §6.2.
We did two different things today.

First we talked about the geometry of eigenvalues and eigenvectors:

- Eigenvectors are vectors \( \mathbf{v} \) such that \( \mathbf{v} \) and \( A\mathbf{v} \) are on the same line through the origin.
- You can pick out the eigenvectors geometrically if you have a picture of the associated transformation.

Then we talked about characteristic polynomials:

- We learned to find the eigenvalues of a matrix by computing the roots of the characteristic polynomial \( p(\lambda) = \det(A - \lambda I) \).
- For a \( 2 \times 2 \) matrix \( A \), the characteristic polynomial is just

\[
p(\lambda) = \lambda^2 - \text{Tr}(A)\lambda + \det(A).
\]

- The algebraic multiplicity of an eigenvalue is its multiplicity as a root of the characteristic polynomial.