Announcements
Wednesday, October 31

- WeBWorK on determinents due today at 11:59pm.

- The quiz on Friday covers §§5.1, 5.2, 5.3.

- My office is Skiles 244 and Rabinoffice hours are: Mondays, 12–1pm; Wednesdays, 1–3pm.
**Definition**

Let $A$ be an $n \times n$ matrix.

1. **An eigenvector** of $A$ is a nonzero vector $v$ in $\mathbb{R}^n$ such that $Av = \lambda v$, for some $\lambda$ in $\mathbb{R}$.

2. **An eigenvalue** of $A$ is a number $\lambda$ in $\mathbb{R}$ such that the equation $Av = \lambda v$ has a nontrivial solution.

3. If $\lambda$ is an eigenvalue of $A$, the $\lambda$-**eigenspace** is the solution set of $(A - \lambda I_n)x = 0$. 


An eigenvector of a matrix $A$ is a nonzero vector $v$ such that:

- $Av$ is a multiple of $v$, which means
- $Av$ is collinear with $v$, which means
- $Av$ and $v$ are on the same line through the origin.

$v$ is an eigenvector

$w$ is not an eigenvector
Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be reflection over the line $L$ defined by $y = -x$, and let $A$ be the matrix for $T$.

**Question:** What are the eigenvalues and eigenspaces of $A$? No computations!

Does anyone see any eigenvectors (vectors that don’t move off their line)?

$v$ is an eigenvector with eigenvalue $-1$. 

[interactive]
Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be reflection over the line $L$ defined by $y = -x$, and let $A$ be the matrix for $T$.

**Question:** What are the eigenvalues and eigenspaces of $A$? No computations!

Does anyone see any eigenvectors (vectors that don’t move off their line)?

$w$ is an eigenvector with eigenvalue 1.
Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be reflection over the line $L$ defined by $y = -x$, and let $A$ be the matrix for $T$.

**Question:** What are the eigenvalues and eigenspaces of $A$? No computations!

Does anyone see any eigenvectors (vectors that don’t move off their line)?

$u$ is *not* an eigenvector.
Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be reflection over the line $L$ defined by $y = -x$, and let $A$ be the matrix for $T$.

**Question:** What are the eigenvalues and eigenspaces of $A$? No computations!

Does anyone see any eigenvectors (vectors that don’t move off their line)? Neither is $z$. 
Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be reflection over the line $L$ defined by $y = -x$, and let $A$ be the matrix for $T$.

**Question:** What are the eigenvalues and eigenspaces of $A$? No computations!

Does anyone see any eigenvectors (vectors that don't move off their line)?

The 1-eigenspace is $L$ (all the vectors $x$ where $Ax = x$).
Let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be reflection over the line $L$ defined by $y = -x$, and let $A$ be the matrix for $T$.

**Question:** What are the eigenvalues and eigenspaces of $A$? No computations!

Does anyone see any eigenvectors (vectors that don’t move off their line)?

The $(-1)$-eigenspace is the line $y = x$ (all the vectors $x$ where $Ax = -x$).
Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the vertical projection onto the $x$-axis, and let $A$ be the matrix for $T$.

**Question:** What are the eigenvalues and eigenspaces of $A$? No computations!

Does anyone see any eigenvectors (vectors that don’t move off their line)?

$v$ is an eigenvector with eigenvalue 0.
Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the vertical projection onto the $x$-axis, and let $A$ be the matrix for $T$.

**Question:** What are the eigenvalues and eigenspaces of $A$? No computations!

Does anyone see any eigenvectors (vectors that don’t move off their line)?

$w$ is an eigenvector with eigenvalue 1.
Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the vertical projection onto the $x$-axis, and let $A$ be the matrix for $T$.

**Question:** What are the eigenvalues and eigenspaces of $A$? No computations!

Does anyone see any eigenvectors (vectors that don’t move off their line)?

$u$ is *not* an eigenvector.
Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the vertical projection onto the $x$-axis, and let $A$ be the matrix for $T$.

**Question:** What are the eigenvalues and eigenspaces of $A$? No computations!

Does anyone see any eigenvectors (vectors that don’t move off their line)?

Neither is $z$. 

[interactive]
Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the vertical projection onto the $x$-axis, and let $A$ be the matrix for $T$.

**Question:** What are the eigenvalues and eigenspaces of $A$? No computations!

Does anyone see any eigenvectors (vectors that don’t move off their line)?

The 1-eigenspace is the $x$-axis (all the vectors $x$ where $Ax = x$).

[interactive]
Let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the vertical projection onto the $x$-axis, and let $A$ be the matrix for $T$.

**Question:** What are the eigenvalues and eigenspaces of $A$? No computations!

Does anyone see any eigenvectors (vectors that don’t move off their line)?

The 0-eigenspace is **the $y$-axis** (all the vectors $x$ where $Ax = 0x$).
Let
\[
A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},
\]
so \(T(x) = Ax\) is a shear in the \(x\)-direction.

**Question:** What are the eigenvalues and eigenspaces of \(A\)? No computations!

Does anyone see any eigenvectors (vectors that don’t move off their line)?

Vectors \(v\) above the \(x\)-axis are moved right but not up... so they’re not eigenvectors.
Eigenspaces
Geometry; example

Let

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$,

so $T(x) = Ax$ is a shear in the $x$-direction.

**Question:** What are the eigenvalues and eigenspaces of $A$? No computations!

Does anyone see any eigenvectors (vectors that don’t move off their line)?

Vectors $w$ below the $x$-axis are moved left but not down... so they’re not eigenvectors.

[interactive]
Let

\[ A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \]

so \( T(x) = Ax \) is a shear in the \( x \)-direction.

**Question:** What are the eigenvalues and eigenspaces of \( A \)? No computations!

Does anyone see any eigenvectors (vectors that don’t move off their line)?

\( u \) is an eigenvector with eigenvalue 1.
Eigenspaces
Geometry; example

Let

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

so $T(x) = Ax$ is a shear in the $x$-direction.

**Question:** What are the eigenvalues and eigenspaces of $A$? No computations!

Does anyone see any eigenvectors (vectors that don’t move off their line)?

The 1-eigenspace is the $x$-axis (all the vectors $x$ where $Ax = x$).
Let

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

so \( T(x) = Ax \) is a shear in the \( x \)-direction.

**Question:** What are the eigenvalues and eigenspaces of \( A \)? No computations!

Does anyone see any eigenvectors (vectors that don’t move off their line)?

There are no other eigenvectors.

[interactive]
Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be counterclockwise rotation by $45^\circ$, and let $A$ be the matrix for $T$.

Find an eigenvector of $A$ without doing any computations.

A. Okay.

B. No way. There are no eigenvectors!

Poll Answer: B. No way.
Section 6.2

The Characteristic Polynomial
The Characteristic Polynomial

Let $A$ be a square matrix.

$\lambda$ is an eigenvalue of $A$ $\iff$ $Ax = \lambda x$ has a nontrivial solution

$\iff$ $(A - \lambda I)x = 0$ has a nontrivial solution

$\iff$ $A - \lambda I$ is not invertible

$\iff$ $\det(A - \lambda I) = 0$.

This gives us a way to compute the eigenvalues of $A$.

**Definition**

Let $A$ be a square matrix. The **characteristic polynomial** of $A$ is

$$f(\lambda) = \det(A - \lambda I).$$

The **characteristic equation** of $A$ is the equation

$$f(\lambda) = \det(A - \lambda I) = 0.$$

**Important**

The eigenvalues of $A$ are the roots of the characteristic polynomial $f(\lambda) = \det(A - \lambda I)$. 
The Characteristic Polynomial
Example

**Question:** What are the eigenvalues of

\[ A = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} \]?
The Characteristic Polynomial

Example

Question: What is the characteristic polynomial of

\[ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \]?

Answer:

\[ f(\lambda) = \det(A - \lambda I) = \det \begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix} = (a - \lambda)(d - \lambda) - bc = \lambda^2 - (a + d)\lambda + \det(A). \]

What do you notice about \( f(\lambda) \)?

- The constant term is \( \det(A) \), which is zero if and only if \( \lambda = 0 \) is a root.
- The linear term \(- (a + d)\) is the negative of the sum of the diagonal entries of \( A \).

Definition

The **trace** of a square matrix \( A \) is \( \text{Tr}(A) = \text{sum of the diagonal entries of } A \).

Shortcut

The characteristic polynomial of a \( 2 \times 2 \) matrix \( A \) is

\[ f(\lambda) = \lambda^2 - \text{Tr}(A)\lambda + \det(A). \]
Question: What are the eigenvalues of the rabbit population matrix

\[ A = \begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \]?
Algebraic Multiplicity

Definition

The *(algebraic) multiplicity* of an eigenvalue $\lambda$ is its multiplicity as a root of the characteristic polynomial.

This is not a very interesting notion *yet*. It will become interesting when we also define *geometric* multiplicity later.

Example

In the rabbit population matrix, $f(\lambda) = -(\lambda - 2)(\lambda + 1)^2$, so the algebraic multiplicity of the eigenvalue 2 is 1, and the algebraic multiplicity of the eigenvalue $-1$ is 2.

Example

In the matrix $\begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$, $f(\lambda) = (\lambda - (3 - 2\sqrt{2}))(\lambda - (3 + 2\sqrt{2}))$, so the algebraic multiplicity of $3 + 2\sqrt{2}$ is 1, and the algebraic multiplicity of $3 - 2\sqrt{2}$ is 1.
Fact: If $A$ is an $n \times n$ matrix, the characteristic polynomial

$$f(\lambda) = \det(A - \lambda I)$$

turns out to be a polynomial of degree $n$, and its roots are the eigenvalues of $A$:

$$f(\lambda) = (-1)^n \lambda^n + a_{n-1} \lambda^{n-1} + a_{n-2} \lambda^{n-2} + \cdots + a_1 \lambda + a_0.$$
Factoring the Characteristic Polynomial

It’s easy to factor quadratic polynomials:

\[ x^2 + bx + c = 0 \implies x = \frac{-b \pm \sqrt{b^2 - 4c}}{2}. \]

It’s less easy to factor cubics, quartics, and so on:

\[ x^3 + bx^2 + cx + d = 0 \implies x = ??? \]
\[ x^4 + bx^3 + cx^2 + dx + e = 0 \implies x = ??? \]

Read about factoring polynomials by hand in §6.2.
We did two different things today.

First we talked about the geometry of eigenvalues and eigenvectors:

- Eigenvectors are vectors $v$ such that $v$ and $Av$ are on the same line through the origin.
- You can pick out the eigenvectors geometrically if you have a picture of the associated transformation.

Then we talked about characteristic polynomials:

- We learned to find the eigenvalues of a matrix by computing the roots of the characteristic polynomial $p(\lambda) = \det(A - \lambda I)$.
- For a $2 \times 2$ matrix $A$, the characteristic polynomial is just
  \[ p(\lambda) = \lambda^2 - \text{Tr}(A)\lambda + \det(A). \]
- The **algebraic multiplicity** of an eigenvalue is its multiplicity as a root of the characteristic polynomial.