WeBWorK on determinents due on Wednesday at 11:59pm.

The quiz on Friday covers §§5.1, 5.2, 5.3.

My office is Skiles 244 and Rabin office hours are: Mondays, 12–1pm; Wednesdays, 1–3pm.
Chapter 6

Eigenvalues and Eigenvectors
Section 6.1

Eigenvalues and Eigenvectors
A Biology Question

**Motivation**

In a population of rabbits:

1. half of the newborn rabbits survive their first year;
2. of those, half survive their second year;
3. their maximum life span is three years;
4. rabbits have 0, 6, 8 baby rabbits in their three years, respectively.

If you know the population one year, what is the population the next year?

- $f_n = \text{first-year rabbits in year } n$
- $s_n = \text{second-year rabbits in year } n$
- $t_n = \text{third-year rabbits in year } n$

The rules say:

$$\begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} f_n \\ s_n \\ t_n \end{pmatrix} = \begin{pmatrix} f_{n+1} \\ s_{n+1} \\ t_{n+1} \end{pmatrix}.$$

Let $A = \begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix}$ and $v_n = \begin{pmatrix} f_n \\ s_n \\ t_n \end{pmatrix}$. Then $Av_n = v_{n+1}$.  

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**difference equation**
If you know $v_0$, what is $v_{10}$?

$$v_{10} = Av_9 = AAv_8 = \cdots = A^{10}v_0.$$  

This makes it easy to compute examples by computer: [interactive]

<table>
<thead>
<tr>
<th>$v_0$</th>
<th>$v_{10}$</th>
<th>$v_{11}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>30189</td>
<td>61316</td>
</tr>
<tr>
<td>7</td>
<td>7761</td>
<td>15095</td>
</tr>
<tr>
<td>9</td>
<td>1844</td>
<td>3881</td>
</tr>
<tr>
<td>1</td>
<td>9459</td>
<td>19222</td>
</tr>
<tr>
<td>2</td>
<td>2434</td>
<td>4729</td>
</tr>
<tr>
<td>3</td>
<td>577</td>
<td>1217</td>
</tr>
<tr>
<td>4</td>
<td>28856</td>
<td>58550</td>
</tr>
<tr>
<td>7</td>
<td>7405</td>
<td>14428</td>
</tr>
<tr>
<td>8</td>
<td>1765</td>
<td>3703</td>
</tr>
</tbody>
</table>

What do you notice about these numbers?

1. Eventually, each segment of the population doubles every year: $Av_n = v_{n+1} = 2v_n$.

2. The ratios get close to $(16 : 4 : 1)$:

$$v_n = (\text{scalar}) \cdot \begin{pmatrix} 16 \\ 4 \\ 1 \end{pmatrix}.$$  

Translation: 2 is an eigenvalue, and $\begin{pmatrix} 16 \\ 4 \\ 1 \end{pmatrix}$ is an eigenvector!
Eigenvectors and Eigenvalues

Definition
Let $A$ be an $n \times n$ matrix.

Eigenvalues and eigenvectors are only for square matrices.

1. An **eigenvector** of $A$ is a nonzero vector $v$ in $\mathbb{R}^n$ such that $Av = \lambda v$, for some $\lambda$ in $\mathbb{R}$. In other words, $Av$ is a multiple of $v$.

2. An **eigenvalue** of $A$ is a number $\lambda$ in $\mathbb{R}$ such that the equation $Av = \lambda v$ has a nontrivial solution.

If $Av = \lambda v$ for $v \neq 0$, we say $\lambda$ is the **eigenvalue for** $v$, and $v$ is an **eigenvector for** $\lambda$.

Note: Eigenvectors are by definition nonzero. Eigenvalues may be equal to zero.

This is the most important definition in the course.
Verifying Eigenvectors

Example

\[ A = \begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \quad v = \begin{pmatrix} 16 \\ 4 \\ 1 \end{pmatrix} \]

Multiply:

\[ Av = \]

Hence \( v \) is an eigenvector of \( A \), with eigenvalue \( \lambda = 2 \).

Example

\[ A = \begin{pmatrix} 2 & 2 \\ -4 & 8 \end{pmatrix} \quad v = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \]

Multiply:

\[ Av = \]

Hence \( v \) is an eigenvector of \( A \), with eigenvalue \( \lambda = 4 \).
Poll

Which of the vectors

A. \((1,1)\)
B. \((1,-1)\)
C. \((-1,1)\)
D. \((2,1)\)
E. \((0,0)\)

are eigenvectors of the matrix

\[
\begin{pmatrix}
1 & 1 \\
1 & 1 \\
\end{pmatrix}
\]

What are the eigenvalues?

\[
\begin{pmatrix}
1 & 1 \\
1 & 1 \\
\end{pmatrix}
\begin{pmatrix}
1 \\
1 \\
\end{pmatrix} = 2 \begin{pmatrix}
1 \\
1 \\
\end{pmatrix}
\]
eigenvector with eigenvalue 2

\[
\begin{pmatrix}
1 & 1 \\
1 & 1 \\
\end{pmatrix}
\begin{pmatrix}
1 \\
-1 \\
\end{pmatrix} = 0 \begin{pmatrix}
1 \\
-1 \\
\end{pmatrix}
\]
eigenvector with eigenvalue 0

\[
\begin{pmatrix}
1 & 1 \\
1 & 1 \\
\end{pmatrix}
\begin{pmatrix}
-1 \\
1 \\
\end{pmatrix} = 0 \begin{pmatrix}
-1 \\
1 \\
\end{pmatrix}
\]
eigenvector with eigenvalue 0

\[
\begin{pmatrix}
1 & 1 \\
1 & 1 \\
\end{pmatrix}
\begin{pmatrix}
2 \\
1 \\
\end{pmatrix} = \begin{pmatrix}
3 \\
3 \\
\end{pmatrix}
\]
not an eigenvector

\[
\begin{pmatrix}
0 & 0 \\
0 & 0 \\
\end{pmatrix}
\]
is never an eigenvector
Verifying Eigenvalues

**Question:** Is $\lambda = 3$ an eigenvalue of $A = \begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix}$?

In other words, does $Av = 3v$ have a nontrivial solution?

... does $Av - 3v = 0$ have a nontrivial solution?

... does $(A - 3I)v = 0$ have a nontrivial solution?

We know how to answer that! Row reduction!

$A - 3I =$
**Eigenspaces**

**Definition**

Let $A$ be an $n \times n$ matrix and let $\lambda$ be an eigenvalue of $A$. The $\lambda$-eigenspace of $A$ is the set of all eigenvectors of $A$ with eigenvalue $\lambda$, plus the zero vector:

\[
\lambda\text{-eigenspace} = \{ v \text{ in } \mathbb{R}^n \mid Av = \lambda v \} \\
= \{ v \text{ in } \mathbb{R}^n \mid (A - \lambda I)v = 0 \} \\
= \text{Nul}(A - \lambda I).
\]

Since the $\lambda$-eigenspace is a null space, it is a *subspace* of $\mathbb{R}^n$.

How do you find a basis for the $\lambda$-eigenspace? Parametric vector form!
Find a basis for the 3-eigenspace of

\[ A = \begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix}. \]
Eigenspaces

Example

Find a basis for the 2-eigenspace of

\[
A = \begin{pmatrix}
\frac{7}{2} & 0 & 3 \\
-\frac{3}{2} & 2 & -3 \\
-\frac{3}{2} & 0 & -1
\end{pmatrix}.
\]
Find a basis for the $\frac{1}{2}$-eigenspace of

$$A = \begin{pmatrix} 7/2 & 0 & 3 \\ -3/2 & 2 & -3 \\ -3/2 & 0 & -1 \end{pmatrix}.$$
Eigenspaces

Example: picture

\[ A = \begin{pmatrix} 7/2 & 0 & 3 \\ -3/2 & 2 & -3 \\ -3/2 & 0 & -1 \end{pmatrix}. \]

We computed bases for the 2-eigenspace and the 1/2-eigenspace:

2-eigenspace: \[ \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \right\} \]

1/2-eigenspace: \[ \left\{ \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right\} \]

Hence the 2-eigenspace is a plane and the 1/2-eigenspace is a line.
Let $A$ be an $n \times n$ matrix and let $\lambda$ be a number.

1. $\lambda$ is an eigenvalue of $A$ if and only if $(A - \lambda I)x = 0$ has a nontrivial solution, if and only if $\text{Nul}(A - \lambda I) \neq \{0\}$.

2. In this case, finding a basis for the $\lambda$-eigenspace of $A$ means finding a basis for $\text{Nul}(A - \lambda I)$ as usual, i.e. by finding the parametric vector form for the general solution to $(A - \lambda I)x = 0$.

3. The eigenvectors with eigenvalue $\lambda$ are the nonzero elements of $\text{Nul}(A - \lambda I)$, i.e. the nontrivial solutions to $(A - \lambda I)x = 0$. 
We’ve seen that finding eigenvectors for a given eigenvalue is a row reduction problem.

Finding all of the eigenvalues of a matrix is not a row reduction problem! We'll see how to do it in general next time. For now:

Fact: The eigenvalues of a triangular matrix are the diagonal entries.
Fact: A is invertible if and only if 0 is not an eigenvalue of A.
Fact: If \( v_1, v_2, \ldots, v_k \) are eigenvectors of \( A \) with distinct eigenvalues \( \lambda_1, \ldots, \lambda_k \), then \( \{v_1, v_2, \ldots, v_k\} \) is linearly independent.

Why? If \( k = 2 \), this says \( v_2 \) can’t lie on the line through \( v_1 \). But the line through \( v_1 \) is contained in the \( \lambda_1 \)-eigenspace, and \( v_2 \) does not have eigenvalue \( \lambda_1 \).

In general: see §6.1 (or work it out for yourself; it’s not too hard).

Consequence: An \( n \times n \) matrix has at most \( n \) distinct eigenvalues.
The Invertible Matrix Theorem
Addenda

We have a couple of new ways of saying “$A$ is invertible” now:

The Invertible Matrix Theorem
Let $A$ be a square $n \times n$ matrix, and let $T : \mathbb{R}^n \to \mathbb{R}^n$ be the linear transformation $T(x) = Ax$. The following statements are equivalent.

1. $A$ is invertible.

2. $T$ is invertible.

3. The reduced row echelon form of $A$ is $I_n$.

4. $A$ has $n$ pivots.

5. $Ax = 0$ has no solutions other than the trivial one.

6. $\text{Nul}(A) = \{0\}$.

7. nullity$(A) = 0$.

8. The columns of $A$ are linearly independent.

9. The columns of $A$ form a basis for $\mathbb{R}^n$.

10. $T$ is one-to-one.

11. $Ax = b$ is consistent for all $b$ in $\mathbb{R}^n$.

12. $Ax = b$ has a unique solution for each $b$ in $\mathbb{R}^n$.

13. The columns of $A$ span $\mathbb{R}^n$.

14. $\text{Col} A = \mathbb{R}^m$.

15. $\dim \text{Col} A = m$.

16. rank $A = m$.

17. $T$ is onto.

18. There exists a matrix $B$ such that $AB = I_n$.

19. There exists a matrix $B$ such that $BA = I_n$.

20. The determinant of $A$ is not equal to zero.

21. The number 0 is not an eigenvalue of $A$. 
Summary

- **Eigenvectors** and **eigenvalues** are the most important concepts in this course.
- Eigenvectors are by definition nonzero; eigenvalues may be zero.
- The eigenvalues of a triangular matrix are the diagonal entries.
- A matrix is invertible if and only if zero is not an eigenvalue.
- Eigenvectors with distinct eigenvalues are linearly independent.
- The $\lambda$-eigenspace is the set of all $\lambda$-eigenvectors, plus the zero vector.
- You can compute a basis for the $\lambda$-eigenspace by finding the parametric vector form of the solutions of $(A - \lambda I_n)x = 0$. 