The second midterm is on **Friday, October 19**.
- That is one week from this Friday.
- The exam covers §§3.5, 3.6, 3.7, 3.9, 4.1, 4.2, 4.3, 4.4 (through today’s material).
- WeBWorK 4.2, 4.3 are due today at 11:59pm.
- The quiz on Friday covers §§4.2, 4.3
- You can go to other instructors’ office hours; see Canvas announcements.
Section 4.4
Matrix Multiplication
Recall: we can turn any system of linear equations into a matrix equation

\[ Ax = b. \]

This notation is suggestive. Can we solve the equation by “dividing by A”?

\[ x \equiv \frac{b}{A} \]

**Answer:** Sometimes, but you have to know what you’re doing.

Today we’ll study *matrix algebra*: adding and multiplying matrices.

These are not so hard to do. The important thing to understand today is the relationship between *matrix multiplication* and *composition of transformations*. 
More Notation for Matrices

Let $A$ be an $m \times n$ matrix.

We write $a_{ij}$ for the entry in the $i$th row and the $j$th column. It is called the $ij$th entry of the matrix.

The entries $a_{11}, a_{22}, a_{33}, \ldots$ are the diagonal entries; they form the main diagonal of the matrix.

A diagonal matrix is a square matrix whose only nonzero entries are on the main diagonal.

The $n \times n$ identity matrix $I_n$ is the diagonal matrix with all diagonal entries equal to 1. It is special because $I_n v = v$ for all $v$ in $\mathbb{R}^n$. 

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
The **zero matrix** (of size $m \times n$) is the $m \times n$ matrix $0$ with all zero entries.

The **transpose** of an $m \times n$ matrix $A$ is the $n \times m$ matrix $A^T$ whose rows are the columns of $A$. In other words, the $ij$ entry of $A^T$ is $a_{ji}$.

\[
0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]

\[
A \quad \quad \quad \quad \quad \quad A^T
\]

\[
\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \quad \quad \quad \quad \quad \quad \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{pmatrix}
\]

\[\text{flip}\]
Addition and Scalar Multiplication

You add two matrices component by component, like with vectors.

\[
\begin{pmatrix}
  a_{11} & a_{12} & a_{13} \\
  a_{21} & a_{22} & a_{23}
\end{pmatrix} + \begin{pmatrix}
  b_{11} & b_{12} & b_{13} \\
  b_{21} & b_{22} & b_{23}
\end{pmatrix} = \begin{pmatrix}
  a_{11} + b_{11} & a_{12} + b_{12} & a_{13} + b_{13} \\
  a_{21} + b_{21} & a_{22} + b_{22} & a_{23} + b_{23}
\end{pmatrix}
\]

Note you can only add two matrices of the same size.

You multiply a matrix by a scalar by multiplying each component, like with vectors.

\[
c \begin{pmatrix}
  a_{11} & a_{12} & a_{13} \\
  a_{21} & a_{22} & a_{23}
\end{pmatrix} = \begin{pmatrix}
  ca_{11} & ca_{12} & ca_{13} \\
  ca_{21} & ca_{22} & ca_{23}
\end{pmatrix}.
\]

These satisfy the expected rules, like with vectors:

\[
A + B = B + A \quad (A + B) + C = A + (B + C)
\]

\[
c(A + B) = cA + cB \quad (c + d)A = cA + dA
\]

\[
(cd)A = c(dA) \quad A + 0 = A
\]
Matrix Multiplication

Beware: matrix multiplication is more subtle than addition and scalar multiplication.

Let $A$ be an $m \times n$ matrix and let $B$ be an $n \times p$ matrix with columns $v_1, v_2, \ldots, v_p$:

$$B = \begin{pmatrix} v_1 & v_2 & \cdots & v_p \end{pmatrix}.$$  

The product $AB$ is the $m \times p$ matrix with columns $Av_1, Av_2, \ldots, Av_p$:

$$AB \overset{\text{def}}{=} \begin{pmatrix} Av_1 & Av_2 & \cdots & Av_p \end{pmatrix}.$$  

The equality is a definition.

In order for $Av_1, Av_2, \ldots, Av_p$ to make sense, the number of columns of $A$ has to be the same as the number of rows of $B$. Note the sizes of the product!

Example

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & -3 \\ 2 & -2 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 14 & -10 \\ 32 & -28 \end{pmatrix}$$
The Row-Column Rule for Matrix Multiplication

Recall: A row vector of length $n$ times a column vector of length $n$ is a scalar:

$$
\begin{pmatrix}
    a_1 & \cdots & a_n
\end{pmatrix}
\begin{pmatrix}
    b_1 \\
    \vdots \\
    b_n
\end{pmatrix}
= a_1 b_1 + \cdots + a_n b_n.
$$

Another way of multiplying a matrix by a vector is:

$$Ax = \begin{pmatrix}
    r_1 \\
    \vdots \\
    r_m
\end{pmatrix} x = \begin{pmatrix}
    r_1 x \\
    \vdots \\
    r_m x
\end{pmatrix}.
$$

On the other hand, you multiply two matrices by

$$AB = A \begin{pmatrix}
    c_1 & \cdots & c_p
\end{pmatrix} = \begin{pmatrix}
    Ac_1 & \cdots & Ac_p
\end{pmatrix}.
$$

It follows that

$$AB = \begin{pmatrix}
    r_1 \\
    \vdots \\
    r_m
\end{pmatrix} \begin{pmatrix}
    c_1 & \cdots & c_p
\end{pmatrix}
= \begin{pmatrix}
    r_1 c_1 & r_1 c_2 & \cdots & r_1 c_p \\
    r_2 c_1 & r_2 c_2 & \cdots & r_2 c_p \\
    \vdots & \vdots & \cdots & \vdots \\
    r_m c_1 & r_m c_2 & \cdots & r_m c_p
\end{pmatrix}.$$
The Row-Column Rule for Matrix Multiplication

The $ij$ entry of $C = AB$ is the $i$th row of $A$ times the $j$th column of $B$:

$$c_{ij} = (AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}.$$ 

This is how everybody on the planet actually computes $AB$. Diagram ($AB = C$):

Example

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & -3 \\ 2 & -2 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} 1 \cdot 1 + 2 \cdot 2 + 3 \cdot 3 & \square \\ \square & \square \end{pmatrix} = \begin{pmatrix} 14 & \square \\ \square & \square \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & -3 \\ 2 & -2 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} \square & \square \\ 4 \cdot 1 + 5 \cdot 2 + 6 \cdot 3 & \square \end{pmatrix} = \begin{pmatrix} \square & \square \\ 32 & \square \end{pmatrix}$$
Why is this the correct definition of matrix multiplication?

**Definition**
Let \( T: \mathbb{R}^n \rightarrow \mathbb{R}^m \) and \( U: \mathbb{R}^p \rightarrow \mathbb{R}^n \) be transformations. The composition is the transformation

\[
T \circ U: \mathbb{R}^p \rightarrow \mathbb{R}^m \quad \text{defined by} \quad T \circ U(x) = T(U(x)).
\]

This makes sense because \( U(x) \) (the output of \( U \)) is in \( \mathbb{R}^n \), which is the domain of \( T \) (the inputs of \( T \)).

**Fact:** If \( T \) and \( U \) are linear then so is \( T \circ U \).

**Guess:** If \( A \) is the matrix for \( T \), and \( B \) is the matrix for \( U \), what is the matrix for \( T \circ U \)?
Composition of Linear Transformations

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ and $U : \mathbb{R}^p \to \mathbb{R}^n$ be linear transformations. Let $A$ and $B$ be their matrices:

$$A = \begin{pmatrix} T(e_1) & T(e_2) & \cdots & T(e_n) \end{pmatrix} \quad B = \begin{pmatrix} U(e_1) & U(e_2) & \cdots & U(e_p) \end{pmatrix}$$

**Question**

What is the matrix for $T \circ U$?

We find the matrix for $T \circ U$ by plugging in the unit coordinate vectors:

$$T \circ U(e_1) = T(U(e_1)) = T(Be_1) = A(Be_1) = (AB)e_1.$$  

For any other $i$, the same works:

$$T \circ U(e_i) = T(U(e_i)) = T(Be_i) = A(Be_i) = (AB)e_i.$$  

This says that the $i$th column of the matrix for $T \circ U$ is the $i$th column of $AB$.

The matrix of the composition is the product of the matrices!
We can also add and scalar multiply linear transformations:

\[
T, U : \mathbb{R}^n \rightarrow \mathbb{R}^m \quad \Rightarrow \quad T + U : \mathbb{R}^n \rightarrow \mathbb{R}^m \quad (T + U)(x) = T(x) + U(x).
\]

In other words, add transformations “pointwise”.

\[
T : \mathbb{R}^n \rightarrow \mathbb{R}^m \quad c \text{ in } \mathbb{R} \quad \Rightarrow \quad cT : \mathbb{R}^n \rightarrow \mathbb{R}^m \quad (cT)(x) = c \cdot T(x).
\]

In other words, scalar-multiply a transformation “pointwise”.

If \( T \) has matrix \( A \) and \( U \) has matrix \( B \), then:

- \( T + U \) has matrix \( A + B \).
- \( cT \) has matrix \( cA \).

So, transformation algebra is the same as matrix algebra.
Composition of Linear Transformations

Example

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ and $U : \mathbb{R}^2 \to \mathbb{R}^3$ be the matrix transformations

$$T(x) = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix} x \quad U(x) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} x.$$ 

Then the matrix for $T \circ U$ is

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}$$
Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be rotation by $45^\circ$, and let $U: \mathbb{R}^2 \to \mathbb{R}^2$ scale the $x$-coordinate by 1.5. Let’s compute their standard matrices $A$ and $B$:
Composition of Linear Transformations

Another example, continued

So the matrix $C$ for $T \circ U$ is

$$C = AB = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1.5 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \left( \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1.5 \\ 0 \end{pmatrix} \right) \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1.5 & -1 \\ 1.5 & 1 \end{pmatrix}.$$

Check:  

$T \circ U(e_1) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1.5 \\ 1.5 \end{pmatrix}$

$T \circ U(e_2) = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$\therefore C = \frac{1}{\sqrt{2}} \begin{pmatrix} 1.5 & -1 \\ 1.5 & 1 \end{pmatrix}$
Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be projection onto the $yz$-plane, and let $U : \mathbb{R}^3 \to \mathbb{R}^3$ be reflection over the $xy$-plane. Let’s compute their standard matrices $A$ and $B$:

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$
So the matrix $C$ for $T \circ U$ is

$$C = AB = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

**Check:** we did this last time

[interactive: $e_1$] [interactive: $e_2$] [interactive: $e_3$]
Do there exist \textit{nonzero} matrices $A$ and $B$ with $AB = 0$?

Yes! Here’s an example:

\[
\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} (1 & 0)(0) & (1 & 0)(0) \\ (1 & 0)(1) & (1 & 0)(1) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.
\]
Properties of Matrix Multiplication

Mostly matrix multiplication works like you’d expect. Suppose $A$ has size $m \times n$, and that the other matrices below have the right size to make multiplication work.

\[
\begin{align*}
A(BC) &= (AB)C \\
(B + C)A &= BA + CA \\
c(AB) &= A(cB) \\
AI_n &= A
\end{align*}
\[
\begin{align*}
A(B + C) &= (AB + AC) \\
(B + C)A &= BA + CA \\
c(AB) &= (cA)B \\
l_mA &= A
\end{align*}
\]

Most of these are easy to verify.

**Associativity** is $A(BC) = (AB)C$. It is a pain to verify using the row-column rule! Much easier: use associativity of linear transformations:

\[
S \circ (T \circ U) = (S \circ T) \circ U.
\]

This is a good example of an instance where having a conceptual viewpoint saves you a lot of work.

**Recommended:** Try to verify all of them on your own.
Properties of Matrix Multiplication

Caveats

Warnings!

- $AB$ is usually not equal to $BA$.

\[
\begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
2 & 0 \\
0 & 1
\end{pmatrix}
= \begin{pmatrix}
0 & -1 \\
2 & 0
\end{pmatrix}
\begin{pmatrix}
2 & 0 \\
0 & 1
\end{pmatrix}
= \begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
0 & -2 \\
1 & 0
\end{pmatrix}
\]

In fact, $AB$ may be defined when $BA$ is not.

- $AB = AC$ does not imply $B = C$, even if $A \neq 0$.

\[
\begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
1 & 2 \\
3 & 4
\end{pmatrix}
= \begin{pmatrix}
1 & 2 \\
0 & 0
\end{pmatrix}
= \begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
1 & 2 \\
5 & 6
\end{pmatrix}
\]

- $AB = 0$ does not imply $A = 0$ or $B = 0$.

\[
\begin{pmatrix}
1 & 0 \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 \\
1 & 1
\end{pmatrix}
= \begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix}
\]
Powers of a Matrix

Suppose $A$ is a square matrix.

Then $A \cdot A$ makes sense, and has the same size.

Then $A \cdot (A \cdot A)$ also makes sense and has the same size.

**Definition**

Let $n$ be a positive whole number and let $A$ be a square matrix. The *nth power* of $A$ is the product

$$A^n = A \cdot A \cdot \ldots \cdot A$$

$n$ times

**Example**

\[
A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad A^2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \\
A^3 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \\
\vdots \quad A^n = \begin{pmatrix} 1 & n-1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}
\]
The product of an $m \times n$ matrix and an $n \times p$ matrix is an $m \times p$ matrix. I showed you two ways of computing the product.

Composition of linear transformations corresponds to multiplication of matrices.

You have to be careful when multiplying matrices together, because things like commutativity and cancellation fail.

You can take powers of square matrices.