Please fill out the mid-semester survey under “Quizzes” on Canvas.

WeBWorK 3.7, 3.9, 4.1 are due on Wednesday at 11:59pm.

The quiz on Friday covers §§3.7, 3.9, 4.1.

My office is Skiles 244 and Rabin office hours are: Mondays, 12–1pm; Wednesdays, 1–3pm.

I will have a new baby any day now, if it hasn’t already happened. At that point various other instructors will fill in for me.
Section 4.2

One-to-one and Onto Transformations
Recall: Let $A$ be an $m \times n$ matrix. The **matrix transformation** associated to $A$ is the transformation

$$T : \mathbb{R}^n \rightarrow \mathbb{R}^m \text{ defined by } T(x) = Ax.$$ 

- The *domain* of $T$ is $\mathbb{R}^n$, which is the number of columns of $A$.
- The *codomain* of $T$ is $\mathbb{R}^m$, which is the number of rows of $A$.
- The *range* of $T$ is the set of all images of $T$:

$$T(x) = Ax = \begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1 v_1 + x_2 v_2 + \cdots + x_n v_n.$$ 

This is the *column space* of $A$. It is a span of vectors in the codomain.
Matrix Transformations

Example

Let \( A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \) and let \( T(x) = Ax \), so \( T: \mathbb{R}^2 \rightarrow \mathbb{R}^3 \).

- If \( u = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \) then \( T(u) = \begin{pmatrix} 7 \\ 5 \\ 7 \end{pmatrix} \).

- Let \( b = \begin{pmatrix} 7 \\ 5 \\ 7 \end{pmatrix} \). Find \( v \) in \( \mathbb{R}^2 \) such that \( T(v) = b \). Is there more than one?
Matrix Transformations

Example, continued

Let \( A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \) and let \( T(x) = Ax \), so \( T: \mathbb{R}^2 \rightarrow \mathbb{R}^3 \).

▶ Is there any \( c \) in \( \mathbb{R}^3 \) such that there is more than one \( v \) in \( \mathbb{R}^2 \) with \( T(v) = c \)?

Translation: is there any \( c \) in \( \mathbb{R}^3 \) such that the solution set of \( Ax = c \) has more than one vector \( v \) in it?

The solution set of \( Ax = c \) is a translate of the solution set of \( Ax = b \) (from before), which has one vector in it. So the solution set to \( Ax = c \) has only one vector. So no!

▶ Find \( c \) such that there is no \( v \) with \( T(v) = c \).

Translation: Find \( c \) such that \( Ax = c \) is inconsistent.

Translation: Find \( c \) not in the column space of \( A \) (i.e., the range of \( T \)).

We could draw a picture, or notice that if \( c = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \), then our matrix equation translates into

\[
\begin{align*}
x + y &= 1 \\
y &= 2 \\
x + y &= 3,
\end{align*}
\]

which is obviously inconsistent.
Note: All of these questions are questions about the transformation $T$; it still makes sense to ask them in the absence of the matrix $A$.

The fact that $T$ comes from a matrix means that these questions translate into questions about a matrix, which we know how to do.

Non-example: $T : \mathbb{R}^2 \to \mathbb{R}^3$ \hspace{1cm} $T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \sin x \\ xy \\ \cos y \end{pmatrix}$

Question: Is there any $c$ in $\mathbb{R}^3$ such that there is more than one $\nu$ in $\mathbb{R}^2$ with $T(\nu) = c$?

Note the question still makes sense, although $T$ has no hope of being a matrix transformation.
Questions About Transformations

Today we will focus on two important questions one can ask about a transformation $T : \mathbb{R}^n \to \mathbb{R}^m$:

- Do there exist distinct vectors $x, y$ in $\mathbb{R}^n$ such that $T(x) = T(y)$?

- For every vector $v$ in $\mathbb{R}^m$, does there exist a vector $x$ in $\mathbb{R}^n$ such that $T(x) = v$?

These are subtle because of the multiple quantifiers involved (“for every”, “there exists”).
One-to-one Transformations

Definition
A transformation \( T: \mathbb{R}^n \rightarrow \mathbb{R}^m \) is one-to-one (or into, or injective) if different vectors in \( \mathbb{R}^n \) map to different vectors in \( \mathbb{R}^m \). In other words, for every \( b \) in \( \mathbb{R}^m \), the equation \( T(x) = b \) has at most one solution \( x \). Or, different inputs have different outputs. Note that not one-to-one means at least two different vectors in \( \mathbb{R}^n \) have the same image.
Consider the robot hand transformation from last lecture:

\[(x, y) = f(\theta, \phi, \psi)\]

Define \(f\): \(\mathbb{R}^3 \rightarrow \mathbb{R}^2\) by:

\(f(\theta, \phi, \psi) = \) position of the hand at joint angles \(\theta, \phi, \psi\).

Is \(f\) one-to-one?

Poll

No: there is more than one way to move the hand to the same point.
Characterization of One-to-One Matrix Transformations

**Theorem**
Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a matrix transformation with matrix $A$. Then the following are equivalent:

- $T$ is one-to-one
- $T(x) = b$ has one or zero solutions for every $b$ in $\mathbb{R}^m$
- $Ax = b$ has a unique solution or is inconsistent for every $b$ in $\mathbb{R}^m$
- $Ax = 0$ has a unique solution
- The columns of $A$ are linearly independent
- $A$ has a pivot in every column.

**Question**
If $T: \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one, what can we say about the relative sizes of $n$ and $m$?

**Answer:** $T$ corresponds to an $m \times n$ matrix $A$. In order for $A$ to have a pivot in every column, it must have at least as many rows as columns: $n \leq m$.

$$
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix}
$$

For instance, $\mathbb{R}^3$ is “too big” to map into $\mathbb{R}^2$. 
Define

\[ A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \quad T(x) = Ax, \]

so \( T : \mathbb{R}^2 \rightarrow \mathbb{R}^3 \). Is \( T \) one-to-one?
Define

\[ A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad T(x) = Ax, \]

so \( T: \mathbb{R}^3 \rightarrow \mathbb{R}^2 \). Is \( T \) one-to-one? If not, find two different vectors \( x, y \) such that \( T(x) = T(y) \).
Onto Transformations

**Definition**
A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is onto (or surjective) if the range of $T$ is equal to $\mathbb{R}^m$ (its codomain). In other words, for every $b$ in $\mathbb{R}^m$, the equation $T(x) = b$ has at least one solution. Or, every possible output has an input. Note that *not* onto means there is some $b$ in $\mathbb{R}^m$ which is not the image of any $x$ in $\mathbb{R}^n$. 

[interactive]
Consider the robot hand transformation again:

\[(x, y) = f(\theta, \phi, \psi)\]

Define \(f\): \(\mathbb{R}^3 \rightarrow \mathbb{R}^2\) by:

\[f(\theta, \phi, \psi) = \text{position of the hand at joint angles } \theta, \phi, \psi.\]

Is \(f\) onto?

Poll

No: it can't reach points that are far away.
Theorem
Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a matrix transformation with matrix $A$. Then the following are equivalent:

- $T$ is onto
- $T(x) = b$ has a solution for every $b$ in $\mathbb{R}^m$
- $Ax = b$ is consistent for every $b$ in $\mathbb{R}^m$
- The columns of $A$ span $\mathbb{R}^m$
- $A$ has a pivot in every row

Question
If $T : \mathbb{R}^n \to \mathbb{R}^m$ is onto, what can we say about the relative sizes of $n$ and $m$?

Answer: $T$ corresponds to an $m \times n$ matrix $A$. In order for $A$ to have a pivot in every row, it must have at least as many columns as rows: $m \leq n$.

\[
\begin{pmatrix}
1 & 0 & \ast & 0 & \ast \\
0 & 1 & \ast & 0 & \ast \\
0 & 0 & 0 & 1 & \ast
\end{pmatrix}
\]

For instance, $\mathbb{R}^2$ is “too small” to map onto $\mathbb{R}^3$. 
Define

\[ A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad T(x) = Ax, \]

so \( T: \mathbb{R}^3 \rightarrow \mathbb{R}^2 \). Is \( T \) onto?

Note that \( T \) is onto but not one-to-one.
Define

\[ A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \quad T(x) = Ax, \]

so \( T: \mathbb{R}^2 \to \mathbb{R}^3 \). Is \( T \) onto? If not, find a vector \( v \) in \( \mathbb{R}^3 \) such that there does not exist any \( x \) in \( \mathbb{R}^2 \) with \( T(x) = v \).

The reduced row echelon form of \( A \) is

\[ \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \]

which does not have a pivot in every row. Hence \( A \) is not onto.

In order to find a vector \( v \) not in the range, we notice that \( T(a, b) = (a, b, a) \). In particular, the \( x \)- and \( z \)-coordinates are the same for every vector in the range, so for example, \( v = (1, 2, 3) \) is not in the range.

Note that \( T \) is one-to-one but not onto.
Define

\[ A = \begin{pmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{pmatrix} \quad T(x) = Ax, \]

so \( T : \mathbb{R}^3 \to \mathbb{R}^2 \). Is \( T \) one-to-one? Is it onto?

[interactive]
A transformation \( T \) is **one-to-one** if \( T(x) = b \) has *at most one* solution, for every \( b \) in \( \mathbb{R}^m \).

A transformation \( T \) is **onto** if \( T(x) = b \) has *at least one* solution, for every \( b \) in \( \mathbb{R}^m \).

A matrix transformation with matrix \( A \) is one-to-one if and only if the columns of \( A \) are linearly independent, if and only if \( A \) has a pivot in every column.

A matrix transformation with matrix \( A \) is onto if and only if the columns of \( A \) span \( \mathbb{R}^m \), if and only if \( A \) has a pivot in every row.

Two of the most basic questions one can ask about a transformation is whether it is one-to-one or onto.