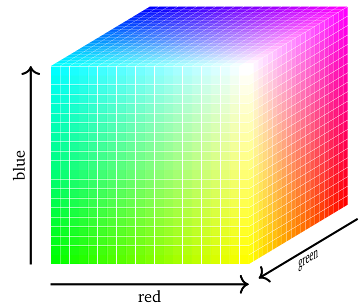
Math 1553 Worksheet §3.5-3.7, 3.9, 4.1

Solutions

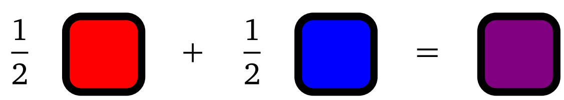
1. Every color on my computer monitor is a vector in $\mathbf{R}^3$ with coordinates between 0 and 255, inclusive. The coordinates correspond to the amount of red, green, and blue in the color.



Given colors $v_1, v_2, \ldots, v_p$, we can form a "weighted average" of these colors by making a linear combination

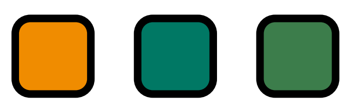
$$v = c_1 v_1 + c_2 v_2 + \cdots + c_p v_p$$

with $c_1 + c_2 + \cdots + c_p = 1$. Example:



Consider the colors on the right. Are these colors linearly independent? What does this tell you about the colors?

$$\begin{pmatrix}240\\140\\0\end{pmatrix} \quad \begin{pmatrix}0\\120\\100\end{pmatrix} \quad \begin{pmatrix}60\\125\\75\end{pmatrix}$$



After doing this problem, check out the interactive demo, where you can adjust sliders to find a prescribed color.

Solution.

The vectors

$$\begin{pmatrix}240\\140\\0\end{pmatrix}, \quad \begin{pmatrix}0\\120\\100\end{pmatrix}, \quad \begin{pmatrix}60\\125\\75\end{pmatrix}$$

are linearly independent if and only if the vector equation

$$x\begin{pmatrix}240\\140\\0\end{pmatrix} + y\begin{pmatrix}0\\120\\100\end{pmatrix} + z\begin{pmatrix}60\\125\\75\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

has only the trivial solution. This translates into the matrix (we don't need to augment since it's a homogeneous system)

$$\begin{pmatrix} 240 & 0 & 60 \\ 140 & 120 & 125 \\ 0 & 100 & 75 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & .25 \\ 0 & 1 & .75 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{\text{parametric}} \begin{aligned} x &= -.25z \\ y &= -.75z \end{aligned}$$

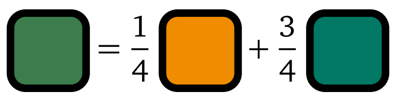
Hence the vectors are linearly *de*pendent; taking $z = 1$ gives the linear dependence relation

$$-\frac{1}{4}\begin{pmatrix} 240 \\ 140 \\ 0 \end{pmatrix} - \frac{3}{4}\begin{pmatrix} 0 \\ 120 \\ 100 \end{pmatrix} + \begin{pmatrix} 60 \\ 125 \\ 75 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Rearranging gives

$$\begin{pmatrix} 60 \\ 125 \\ 75 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 240 \\ 140 \\ 0 \end{pmatrix} + \frac{3}{4}\begin{pmatrix} 0 \\ 120 \\ 100 \end{pmatrix}.$$

In terms of colors:



2. Circle **TRUE** if the statement is always true, and circle **FALSE** otherwise.

a) If A is a 3×100 matrix of rank 2, then $\dim(\mathrm{Nul}\,A) = 97$.

TRUE **FALSE**

b) If A is an $m \times n$ matrix and $Ax = 0$ has only the trivial solution, then the columns of A form a basis for $\mathbf{R}^m$.

TRUE **FALSE**

c) The set $V = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \text{ in } \mathbf{R}^4 \;\middle|\; x - 4z = 0 \right\}$ is a subspace of $\mathbf{R}^4$.

TRUE **FALSE**

Solution.

a) False. By the Rank Theorem, $\mathrm{rank}(A) + \dim(\mathrm{Nul}\,A) = 100$, so $\dim(\mathrm{Nul}\,A) = 98$.

b) False. For example, $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ has only the trivial solution for $Ax = 0$, but its column space is a 2-dimensional subspace of $\mathbf{R}^3$.

c) True. V is $\mathrm{Nul}(A)$ for the 1×4 matrix A below, and therefore is automatically a subspace of $\mathbf{R}^4$:

$$A = \begin{pmatrix} 1 & 0 & -4 & 0 \end{pmatrix}.$$

Alternatively, we could verify the subspace properties directly if we wished. This is much more work!

(1) The zero vector is in V, since $0-4(0)0=0$.

(2) Let $u=\begin{pmatrix}x_1\\y_1\\z_1\\w_1\end{pmatrix}$ and $v=\begin{pmatrix}x_2\\y_2\\z_2\\w_2\end{pmatrix}$ be in V, so $x_1-4z_1=0$ and $x_2-4z_2=0$.
We compute

$$u+v=\begin{pmatrix}x_1+x_2\\y_1+y_2\\z_1+z_2\\w_1+w_2\end{pmatrix}.$$

Is $(x_1+x_2)-4(z_1+z_2)=0$? Yes, since
$(x_1+x_2)-4(z_1+z_2)=(x_1-4z_1)\;+\;(x_2-4z_2)=0+0=0.$

(3) If $u=\begin{pmatrix}x\\y\\z\\w\end{pmatrix}$ is in V then so is cu for any scalar c:

$$cu=\begin{pmatrix}cx\\cy\\cz\\cw\end{pmatrix}\qquad\text{and}\qquad cx-4cz=c(x-4z)=c(0)=0.$$

3. Let $A=\begin{pmatrix}1&-5&-2&-4\\2&3&9&5\\1&1&4&2\end{pmatrix}$, and let T be the matrix transformation associated to A, so $T(x)=Ax$.

a) What is the domain of T? What is the codomain of T? Give an example of a vector in the range of T.

b) The RREF of A is $\begin{pmatrix}1&0&3&1\\0&1&1&1\\0&0&0&0\end{pmatrix}$. Is there a vector in the codomain of T which is not in the range of T? Justify your answer.

Solution.

a) The domain is $\mathbf{R}^4$; the codomain is $\mathbf{R}^3$. The vector $0=T(0)$ is contained in the range, as is

$$\begin{pmatrix}1\\2\\1\end{pmatrix}=T\begin{pmatrix}1\\0\\0\\0\end{pmatrix}.$$

b) Yes. The range of T is the column span of A, and from the RREF of A we know A only has two pivots, so its column span is a 2-dimensional subspace of $\mathbf{R}^3$. Since $\dim(\mathbf{R}^3)=3$, the range is not equal to $\mathbf{R}^3$.