Announcements
Wednesday, September 19

▶ WeBWorK 3.3, 3.4 are due today at 11:59pm.
▶ The first midterm is on this Friday, September 21.
  ▶ Midterms happen during recitation.
  ▶ The exam covers through §3.4.
  ▶ About half the problems will be conceptual, and the other half computational.
▶ There is a practice midterm posted on the website. It is meant to be similar in format and difficulty to the real midterm. Solutions are posted.
▶ Study tips:
  ▶ Drill problems in Lay. Practice the recipes until you can do them in your sleep.
  ▶ Make sure to learn the theorems and learn the definitions, and understand what they mean. There is a reference sheet on the website. Make flashcards!
  ▶ Sit down to do the practice midterm in 50 minutes, with no notes.
  ▶ Come to office hours!
▶ Double Rabinoffice hours this week: Monday 12–1; Tuesday 10–11; Wednesday 1–3; Thursday 2–4
▶ TA review session: Weber SST III classroom 1, 4:30–6pm on Thursday.
Section 3.6

Subspaces
Today we will discuss **subspaces** of $\mathbb{R}^n$.

A subspace turns out to be the same as a span, except we don’t know *which* vectors it’s the span of.

This arises naturally when you have, say, a plane through the origin in $\mathbb{R}^3$ which is *not* defined (a priori) as a span, but you still want to say something about it.

\[ x + 3y + z = 0 \]
Definition

A **subspace** of $\mathbb{R}^n$ is a subset $V$ of $\mathbb{R}^n$ satisfying:

1. The zero vector is in $V$. "not empty"
2. If $u$ and $v$ are in $V$, then $u + v$ is also in $V$. "closed under addition"
3. If $u$ is in $V$ and $c$ is in $\mathbb{R}$, then $cu$ is in $V$. "closed under $\times$ scalars"

Every subspace is a span, and every span is a subspace.

A subspace is a span of some vectors, but you haven’t computed what those vectors are yet.
Definition of Subspace

Definition
A **subspace** of \( \mathbb{R}^n \) is a subset \( V \) of \( \mathbb{R}^n \) satisfying:

1. The zero vector is in \( V \).  
   **“not empty”**
2. If \( u \) and \( v \) are in \( V \), then \( u + v \) is also in \( V \).  
   **“closed under addition”**
3. If \( u \) is in \( V \) and \( c \) is in \( \mathbb{R} \), then \( cu \) is in \( V \).  
   **“closed under \( \times \) scalars”**

What does this mean?

- If \( v \) is in \( V \), then all scalar multiples of \( v \) are in \( V \) by (3). That is, the line through \( v \) is in \( V \).

- If \( u, v \) are in \( V \), then \( xu \) and \( yv \) are in \( V \) for scalars \( x, y \) by (3). So \( xu + yv \) is in \( V \) by (2). So Span\{\( u, v \)\} is contained in \( V \).

- Likewise, if \( v_1, v_2, \ldots, v_n \) are all in \( V \), then Span\{\( v_1, v_2, \ldots, v_n \)\} is contained in \( V \): a subspace contains the span of any set of vectors in it.

If you pick enough vectors in \( V \), eventually their span will fill up \( V \), so:

A subspace is a span of some set of vectors in it.
Examples

Example
A line $L$ through the origin: this contains the span of any vector in $L$.

Example
A plane $P$ through the origin: this contains the span of any vectors in $P$.

Example
All of $\mathbb{R}^n$: this contains 0, and is closed under addition and scalar multiplication.

Example
The subset \{0\}: this subspace contains only one vector.

Note these are all pictures of spans! (Line, plane, space, etc.)
A **subset** of $\mathbb{R}^n$ is any collection of vectors whatsoever.

All of the following non-examples are still subsets.

A **subspace** is a special kind of subset, which satisfies the three defining properties.

- Subset: *yes*
- Subspace: *no*
Non-Examples

Non-Example
A line $L$ (or any other set) that doesn't contain the origin is not a subspace. Fails: 1.

Non-Example
A circle $C$ is not a subspace. Fails: 1,2,3. Think: a circle isn't a "linear space."

Non-Example
The first quadrant in $\mathbb{R}^2$ is not a subspace. Fails: 3 only.

Non-Example
A line union a plane in $\mathbb{R}^3$ is not a subspace. Fails: 2 only.
Theorem
Any \( \text{Span}\{v_1, v_2, \ldots, v_n\} \) is a subspace.

Every subspace is a span, and every span is a subspace.

Definition
If \( V = \text{Span}\{v_1, v_2, \ldots, v_n\} \), we say that \( V \) is the subspace generated by or spanned by the vectors \( v_1, v_2, \ldots, v_n \).
Poll

Which of the following are subspaces?

A. The empty set
B. The solution set to a homogeneous system of linear equations.
C. The solution set to an inhomogeneous system of linear equations.
D. The set of all vectors in \( \mathbb{R}^n \) with rational (fraction) coordinates.

For the ones which are not subspaces, which property(ies) do they not satisfy?

A. This is not a subspace: it does not contain the zero vector.
B. This is a subspace: the solution set is a span, produced by finding the parametric vector form of the solution.
C. This is not a subspace: it does not contain 0.
D. This is not a subspace: it is not closed under multiplication by scalars (e.g. by \( \pi \)).
Let \( V = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2 \mid ab = 0 \right\} \). Let’s check if \( V \) is a subspace or not.

1. Does \( V \) contain the zero vector?

\[ \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow ab = 0 \]

3. Is \( V \) closed under scalar multiplication?

Let \( \begin{pmatrix} a \\ b \end{pmatrix} \) be (an unknown vector) in \( V \).

This means: \( a \) and \( b \) are numbers such that \( ab = 0 \).

Let \( c \) be a scalar. Is \( c \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} ca \\ cb \end{pmatrix} \) in \( V \)?

This means: \( (ca)(cb) = 0 \).

\[ (ca)(cb) = c^2(ab) = c^2(0) = 0 \]

2. Is \( V \) closed under addition?

Let \( \begin{pmatrix} a \\ b \end{pmatrix} \) and \( \begin{pmatrix} a' \\ b' \end{pmatrix} \) be (unknown vectors) in \( V \).

This means: \( ab = 0 \) and \( a'b' = 0 \).

Is \( \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} a' \\ b' \end{pmatrix} = \begin{pmatrix} a + a' \\ b + b' \end{pmatrix} \) in \( V \)?

This means: \( (a + a')(b + b') = 0 \).

This is not true for all such \( a, a', b, b' \): for instance, \( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) and \( \begin{pmatrix} 0 \\ 1 \end{pmatrix} \) are in \( V \), but their sum \( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \) is not in \( V \), because \( 1 \cdot 1 \neq 0 \).

We conclude that \( V \) is not a subspace. A picture is above. (It doesn’t look like a span.)
An $m \times n$ matrix $A$ naturally gives rise to two subspaces.

**Definition**

- The **column space** of $A$ is the subspace of $\mathbf{R}^m$ spanned by the columns of $A$. It is written $\text{Col} \ A$.

- The **null space** of $A$ is the set of all solutions of the homogeneous equation $Ax = 0$:

$$\text{Nul} \ A = \{ \mathbf{x} \in \mathbf{R}^n \mid Ax = 0 \}.$$  

This is a subspace of $\mathbf{R}^n$.

The column space is defined as a span, so we know it is a subspace.

**Check** that the null space is a subspace:
Column Space and Null Space

Example

Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}$.

Let’s compute the column space:

Let’s compute the null space:
The Null Space is a Span

The column space of a matrix $A$ is defined to be a span (of the columns). The null space is defined to be the solution set to $Ax = 0$. It is a subspace, so it is a span.

**Question**
How to find vectors which span the null space?

**Answer:** Parametric vector form! We know that the solution set to $Ax = 0$ has a parametric form that looks like

$$x_3 \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -2 \\ 3 \\ 0 \\ 1 \end{pmatrix}$$

if, say, $x_3$ and $x_4$ are the free variables. So

$$\text{Nul } A = \text{Span} \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 3 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Refer back to the slides for §3.4 (Solution Sets).

**Note:** It is much easier to define the null space first as a subspace, then find spanning vectors *later*, if we need them. This is one reason subspaces are so useful.
Subspaces

Summary

- A **subspace** is the same as a span of some number of vectors, but we haven’t computed the vectors yet.
- To any matrix is associated two subspaces, the **column space** and the **null space**:
  
  $\text{Col } A = \text{the span of the columns of } A$

  $\text{Nul } A = \text{the solution set of } Ax = 0.$

**How do you check if a subset is a subspace?**

- Is it a span? Can it be written as a span?
- Can it be written as the column space of a matrix?
- Can it be written as the null space of a matrix?
- Is it all of $\mathbb{R}^n$ or the zero subspace $\{0\}$?
- Can it be written as a type of subspace that we’ll learn about later (eigenspaces, . . . )?

If so, then it’s automatically a subspace.

If all else fails:

- Can you verify directly that it satisfies the three defining properties?