Announcements

Wednesday, September 12

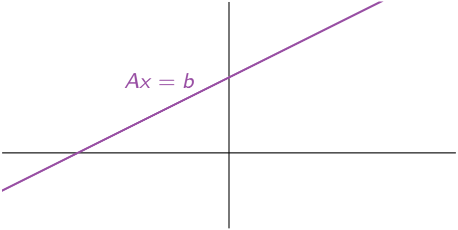
- WeBWorK 3.1, 3.2 due today at 11:59pm.
- The quiz on Friday covers §3.1 and §3.2.
- The first midterm is on **Friday, September 21**.
  - That is one week from this Friday.
  - Midterms happen during recitation.
  - The exam covers *through §3.4* (today's material).
- My office is Skiles 244 and Rabinoffice hours are: Mondays, 12–1pm; Wednesdays, 1–3pm.

Section 3.4

Solution Sets

Plan For Today

Today we will learn to describe and draw the solution set of an arbitrary system of linear equations $Ax = b$, using spans.



Recall: the **solution set** is the collection of all vectors x such that $Ax = b$ is true.

Last time we discussed the set of vectors b for which $Ax = b$ has a solution.

We also described this set using spans, but it was a *different problem*.

Homogeneous Systems

Everything is easier when $b = 0$, so we start with this case.

Definition

A system of linear equations of the form $Ax = 0$ is called **homogeneous.**

These are linear equations where everything to the right of the $=$ is zero. The opposite is:

Definition

A system of linear equations of the form $Ax = b$ with $b \neq 0$ is called **inhomogeneous.**

A homogeneous system always has the solution $x = 0$. This is called the **trivial solution.** The nonzero solutions are called **nontrivial.**

Observation

$Ax = 0$ has a nontrivial solution

$\iff$ there is a free variable

$\iff$ A has a column with no pivot.

Question

What is the solution set of $Ax = 0$, where

$$A = \begin{pmatrix} 1 & 3 & 4 \\ 2 & -1 & 2 \\ 1 & 0 & 1 \end{pmatrix}?$$

We know how to do this: first form an augmented matrix and row reduce.

The only solution is the trivial solution $x = 0$.

Observation

Since the last column (everything to the right of the $=$) was zero to begin, it will always stay zero! So it's not really necessary to write augmented matrices in the homogeneous case.

Question

What is the solution set of $Ax = 0$, where

$$A = \begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix}?$$

This last equation is called the **parametric vector form** of the solution.

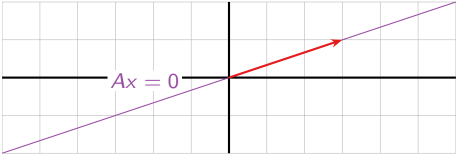
It is obtained by listing equations for all the variables, in order, including the free ones, and making a vector equation.

Question

What is the solution set of $Ax = 0$, where

$$A = \begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix}?$$

Answer: $x = x_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ for any x_2 in $\mathbf{R}$. The solution set is $\text{Span}\left\{ \begin{pmatrix} 3 \\ 1 \end{pmatrix} \right\}$.



[interactive]

Note: *one* free variable means the solution set is a *line* in $\mathbf{R}^2$ ($2 = \#$ variables $= \#$ columns).

Question

What is the solution set of $Ax = 0$, where

$$A = \begin{pmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{pmatrix}?$$

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What is the solution set of $Ax = 0$, where

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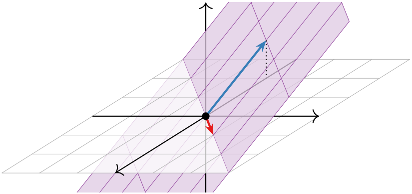
Answer: Span $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \right\}$.



[interactive]

Note: *two* free variables means the solution set is a *plane* in $\mathbf{R}^3$ ($3 = \#$ variables $= \#$ columns).

Question

What is the solution set of $Ax = 0$, where $A =$

$$\begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix} \xrightarrow{\text{row reduce}} \begin{pmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{\text{equations}} \begin{cases} x_1 \qquad - 8x_3 - 7x_4 = 0 \\ \qquad x_2 + 4x_3 + 3x_4 = 0 \end{cases}$$

$$\xrightarrow{\text{parametric form}} \begin{cases} x_1 = 8x_3 + 7x_4 \\ x_2 = -4x_3 - 3x_4 \\ x_3 = x_3 \\ x_4 = x_4 \end{cases}$$

$$\xrightarrow{\text{parametric vector form}} x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix}.$$

Question

What is the solution set of $Ax = 0$, where

$$A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix}?$$

Answer: $\text{Span}\left\{ \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix} \right\}.$

[not pictured here]

Note: *two* free variables means the solution set is a *plane* in $\mathbf{R}^4$ ($4 = \#$ variables $= \#$ columns).

Let A be an $m \times n$ matrix. Suppose that the free variables in the homogeneous equation $Ax = 0$ are $x_i, x_j, x_k, \ldots$

Then the solutions to $Ax = 0$ can be written in the form

$$x = x_i v_i + x_j v_j + x_k v_k + \cdots$$

for some vectors $v_i, v_j, v_k, \ldots$ in $\mathbf{R}^n$, and any scalars $x_i, x_j, x_k, \ldots$

The solution set is

$$\text{Span}\{v_i,\ v_j,\ v_k, \ldots\}.$$

The equation above is called the **parametric vector form** of the solution.

It is obtained by listing equations for all the variables, in order, including the free ones, and making a vector equation.

Poll

Question

What is the solution set of $Ax = b$, where

$$A = \begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} -3 \\ -6 \end{pmatrix}?$$

The only difference from the homogeneous case is the constant vector $p = \binom{-3}{0}$.

Note that p is itself a solution: take $x_2 = 0$.

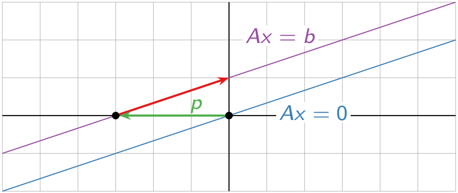
Question

What is the solution set of $Ax = b$, where

$$A = \begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} -3 \\ -6 \end{pmatrix}?$$

Answer: $x = x_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \begin{pmatrix} -3 \\ 0 \end{pmatrix}$ for any x_2 in $\mathbf{R}$.

This is a *translate* of $\text{Span}\left\{\begin{pmatrix} 3 \\ 1 \end{pmatrix}\right\}$: it is the parallel line through $p = \begin{pmatrix} -3 \\ 0 \end{pmatrix}$.



It can be written

$$\text{Span}\left\{\begin{pmatrix} 3 \\ 1 \end{pmatrix}\right\} + \begin{pmatrix} -3 \\ 0 \end{pmatrix}.$$

[interactive]

Question

What is the solution set of $Ax = b$, where

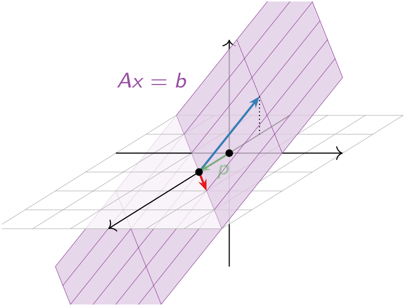
$$A = \begin{pmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 \\ -2 \end{pmatrix}?$$

Question

What is the solution set of $Ax = b$, where

$$A = \begin{pmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 \\ -2 \end{pmatrix}?$$

Answer: $\text{Span}\left\{\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}-2\\0\\1\end{pmatrix}\right\} + \begin{pmatrix}1\\0\\0\end{pmatrix}.$



The solution set is a *translate* of

$$\text{Span}\left\{\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}-2\\0\\1\end{pmatrix}\right\}:$$

it is the parallel plane through

$$p = \begin{pmatrix}1\\0\\0\end{pmatrix}.$$

[interactive]

Homogeneous vs. Inhomogeneous Systems

Key Observation

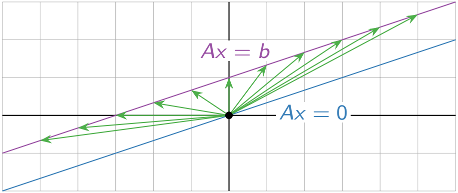
The set of solutions to $Ax = b$, if it is nonempty, is obtained by taking one **specific** or **particular solution** p to $Ax = b$, and adding all solutions to $Ax = 0$.

Why? If $Ap = b$ and $Ax = 0$, then

$$A(p + x) = Ap + Ax = b + 0 = b,$$

so $p + x$ is also a solution to $Ax = b$.

We know the solution set of $Ax = 0$ is a span. So the solution set of $Ax = b$ is a *translate* of a span: it is *parallel* to a span. (Or it is empty.)



This works for *any* specific solution p: it doesn't have to be the one produced by finding the parametric vector form and setting the free variables all to zero, as we did before.

[interactive 1] [interactive 2]

Solution Sets and Spans of Columns

Very Important

Let A be an $m \times n$ matrix. There are now two *completely different* things you know how to describe using spans:

- The **solution set:** for fixed b, this is all x such that $Ax = b$.
  - This is a span if $b = 0$, or a translate of a span in general (if it's consistent).
  - Lives in $\mathbf{R}^n$.
  - Computed by finding the parametric vector form.

- The **span of the columns:** this is all b such that $Ax = b$ is consistent.
  - This is the span of the columns of A.
  - Lives in $\mathbf{R}^m$.

Don't confuse these two geometric objects!

Much of the first midterm tests whether you understand both.

[interactive]

Summary

- The solution set to a **homogeneous** system $Ax = 0$ is a span. It always contains the **trivial solution** $x = 0$.
- The solution set to a **nonhomogeneous** system $Ax = b$ is either empty, or it is a translate of a span: namely, it is a translate of the solution set of $Ax = 0$.
- The solution set to $Ax = b$ can be expressed as a translate of a span by computing the **parametric vector form** of the solution.
- The solution set to $Ax = b$ and the span of the columns of A (from the previous lecture) are two completely different things, and you have to understand them separately.