Announcements
Wednesday, September 05

▶ WeBWorK 2.2, 2.3 due today at 11:59pm.
▶ The quiz on Friday covers through §2.3 (last week’s material).
▶ My office is Skiles 244 and Rabin office hours are: Mondays, 12–1pm; Wednesdays, 1–3pm.
▶ Your TAs have office hours too. You can go to any of them. Details on the website.
Chapter 3

Systems of Linear Equations: Geometry
Motivation

We want to think about the algebra in linear algebra (systems of equations and their solution sets) in terms of geometry (points, lines, planes, etc).

\[
\begin{align*}
x - 3y &= -3 \\
2x + y &= 8
\end{align*}
\]

This will give us better insight into the properties of systems of equations and their solution sets.

Remember: I expect you to be able to draw pictures!
Section 3.1

Vectors
Points and Vectors

We have been drawing elements of \( \mathbb{R}^n \) as points in the line, plane, space, etc. We can also draw them as arrows.

**Definition**

A **point** is an element of \( \mathbb{R}^n \), drawn as a point (a dot).

A **vector** is an element of \( \mathbb{R}^n \), drawn as an arrow. When we think of an element of \( \mathbb{R}^n \) as a vector, we’ll usually write it vertically, like a matrix with one column:

\[
v = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.
\]

The difference is purely psychological: *points and vectors are just lists of numbers.*
So why make the distinction?

A vector need not start at the origin: *it can be located anywhere!* In other words, an arrow is determined by its length and its direction, not by its location.

These arrows all represent the vector \( \begin{pmatrix} 1 \\ 2 \end{pmatrix} \).

However, unless otherwise specified, we’ll assume a vector starts at the origin.
Vector Algebra

Definition

▶ We can add two vectors together:

\[
\begin{pmatrix}
a \\
b \\
c
\end{pmatrix}
+ \begin{pmatrix}
x \\
y \\
z
\end{pmatrix} = \begin{pmatrix}
a + x \\
b + y \\
c + z
\end{pmatrix}.
\]

▶ We can multiply, or \textbf{scale}, a vector by a real number \(c\):

\[
c \begin{pmatrix}
x \\
y \\
z
\end{pmatrix} = \begin{pmatrix}
c \cdot x \\
c \cdot y \\
c \cdot z
\end{pmatrix}.
\]

We call \(c\) a \textbf{scalar} to distinguish it from a vector. If \(v\) is a vector and \(c\) is a scalar, \(cv\) is called a \textbf{scalar multiple} of \(v\).

(And likewise for vectors of length \(n\).) For instance,

\[
\begin{pmatrix}
1 \\
2 \\
3
\end{pmatrix}
+ \begin{pmatrix}
4 \\
5 \\
6
\end{pmatrix} = \begin{pmatrix}
5 \\
7 \\
9
\end{pmatrix}
\quad \text{and} \quad
-2 \begin{pmatrix}
1 \\
2 \\
3
\end{pmatrix} = \begin{pmatrix}
-2 \\
-4 \\
-6
\end{pmatrix}.
\]
Vector Addition and Subtraction: Geometry

The parallelogram law for vector addition

Geometrically, the sum of two vectors \( \mathbf{v}, \mathbf{w} \) is obtained as follows: place the tail of \( \mathbf{w} \) at the head of \( \mathbf{v} \). Then \( \mathbf{v} + \mathbf{w} \) is the vector whose tail is the tail of \( \mathbf{v} \) and whose head is the head of \( \mathbf{w} \). Doing this both ways creates a parallelogram. For example,

\[
\begin{pmatrix} 1 \\ 3 \end{pmatrix} + \begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \end{pmatrix}.
\]

Why? The width of \( \mathbf{v} + \mathbf{w} \) is the sum of the widths, and likewise with the heights. [interactive]

Vector subtraction

Geometrically, the difference of two vectors \( \mathbf{v}, \mathbf{w} \) is obtained as follows: place the tail of \( \mathbf{v} \) and \( \mathbf{w} \) at the same point. Then \( \mathbf{v} - \mathbf{w} \) is the vector from the head of \( \mathbf{w} \) to the head of \( \mathbf{v} \). For example,

\[
\begin{pmatrix} 1 \\ 4 \end{pmatrix} - \begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}.
\]

Why? If you add \( \mathbf{v} - \mathbf{w} \) to \( \mathbf{w} \), you get \( \mathbf{v} \). [interactive]

This works in higher dimensions too!
**Scalar Multiplication: Geometry**

**Scalar multiples of a vector**
These have the same *direction* but a different *length*.

Some multiples of \( \mathbf{v} \).

- \( \mathbf{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \)
- \( 2\mathbf{v} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} \)
- \( -\frac{1}{2}\mathbf{v} = \begin{pmatrix} -\frac{1}{2} \\ -1 \end{pmatrix} \)
- \( 0\mathbf{v} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \)

All multiples of \( \mathbf{v} \).

So the scalar multiples of \( \mathbf{v} \) form a *line*. [interactive]
Linear Combinations

We can add and scalar multiply in the same equation:

\[ w = c_1 v_1 + c_2 v_2 + \cdots + c_p v_p \]

where \( c_1, c_2, \ldots, c_p \) are scalars, \( v_1, v_2, \ldots, v_p \) are vectors in \( \mathbb{R}^n \), and \( w \) is a vector in \( \mathbb{R}^n \).

Definition
We call \( w \) a **linear combination** of the vectors \( v_1, v_2, \ldots, v_p \). The scalars \( c_1, c_2, \ldots, c_p \) are called the **weights** or **coefficients**.

Example

Let \( v = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \) and \( w = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \).

What are some linear combinations of \( v \) and \( w \)?

- \( v + w \)
- \( v - w \)
- \( 2v + 0w \)
- \( 2w \)
- \( -v \)

[interactive: 2 vectors] [interactive: 3 vectors]
Is there any vector in $\mathbb{R}^2$ that is *not* a linear combination of $v$ and $w$?

No: in fact, *every* vector in $\mathbb{R}^2$ is a combination of $v$ and $w$.

(The purple lines are to help measure *how much* of $v$ and $w$ you need to get to a given point.)
What are some linear combinations of \( v = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \)?

\[ \begin{align*}
&\triangleq \frac{3}{2} v \\
&\triangleq -\frac{1}{2} v \\
&\triangleq \ldots
\end{align*} \]

What are all linear combinations of \( v \)?
All vectors \( cv \) for \( c \) a real number. I.e., all scalar multiples of \( v \). These form a line.

**Question**
What are all linear combinations of \( v = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \) and \( w = \begin{pmatrix} -1 \\ -1 \end{pmatrix} \)?

**Answer:** The line which contains both vectors.

What’s different about this example and the one on the poll? [interactive]
Section 3.2

Vector Equations and Spans
Solve the following system of linear equations:

\[
\begin{align*}
    x - y &= 8 \\
    2x - 2y &= 16 \\
    6x - y &= 3.
\end{align*}
\]

We can write all three equations at once as vectors:

\[
\begin{pmatrix}
    x - y \\
    2x - 2y \\
    6x - y
\end{pmatrix}
= \begin{pmatrix}
    8 \\
    16 \\
    3
\end{pmatrix}.
\]

We can write this as a linear combination:

\[
x \begin{pmatrix}
    1 \\
    2 \\
    6
\end{pmatrix}
+ y \begin{pmatrix}
    -1 \\
    -2 \\
    -1
\end{pmatrix}
= \begin{pmatrix}
    8 \\
    16 \\
    3
\end{pmatrix}.
\]

So we are asking:

**Question:** Is \( \begin{pmatrix}
    8 \\
    16 \\
    3
\end{pmatrix} \) a linear combination of \( \begin{pmatrix}
    1 \\
    2 \\
    6
\end{pmatrix} \) and \( \begin{pmatrix}
    -1 \\
    -2 \\
    -1
\end{pmatrix} \)?
Systems of Linear Equations
Continued

\[
x - y = 8 \\
2x - 2y = 16 \\
6x - y = 3
\]

\[
\begin{pmatrix}
1 & -1 & 8 \\
2 & -2 & 16 \\
6 & -1 & 3
\end{pmatrix}
\]

Row reduce
\[
\begin{pmatrix}
1 & 0 & -1 \\
0 & 1 & -9 \\
0 & 0 & 0
\end{pmatrix}
\]

Solution
\[
x = -1 \\
y = -9
\]

Conclusion:
\[
- \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} - 9 \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}
\]

[interactive] ← (this is the picture of a consistent linear system)

What is the relationship between the vectors in the linear combination and the matrix form of the linear equation? They have the same columns!

Shortcut: You can go directly between augmented matrices and vector equations.
Vector Equations and Linear Equations

**Summary**

The **vector equation**

\[ x_1 v_1 + x_2 v_2 + \cdots + x_p v_p = b, \]

where \( v_1, v_2, \ldots, v_p, b \) are vectors in \( \mathbb{R}^n \) and \( x_1, x_2, \ldots, x_p \) are scalars, has the same solution set as the linear system with augmented matrix

\[
\begin{pmatrix}
| & | & | \\
| v_1 & v_2 & \cdots & v_p & b \\
| & | & | \\
\end{pmatrix},
\]

where the \( v_i \)'s and \( b \) are the columns of the matrix.

So we now have (at least) **two** equivalent ways of thinking about linear systems of equations:

1. Augmented matrices.
2. Linear combinations of vectors (vector equations).

The last one is more geometric in nature.
It is important to know what are all linear combinations of a set of vectors $v_1, v_2, \ldots, v_p$ in $\mathbb{R}^n$: it’s exactly the collection of all $b$ in $\mathbb{R}^n$ such that the vector equation (in the unknowns $x_1, x_2, \ldots, x_p$)

$$x_1 v_1 + x_2 v_2 + \cdots + x_p v_p = b$$

has a solution (i.e., is consistent).

**Definition**

Let $v_1, v_2, \ldots, v_p$ be vectors in $\mathbb{R}^n$. The span of $v_1, v_2, \ldots, v_p$ is the collection of all linear combinations of $v_1, v_2, \ldots, v_p$, and is denoted $\text{Span}\{v_1, v_2, \ldots, v_p\}$. In symbols:

$$\text{Span}\{v_1, v_2, \ldots, v_p\} = \{ x_1 v_1 + x_2 v_2 + \cdots + x_p v_p \mid x_1, x_2, \ldots, x_p \text{ in } \mathbb{R} \}.$$ 

**Synonyms:** $\text{Span}\{v_1, v_2, \ldots, v_p\}$ is the subset spanned by or generated by $v_1, v_2, \ldots, v_p$.

This is the first of several definitions in this class that you simply **must learn**. I will give you other ways to think about Span, and ways to draw pictures, but **this is the definition**. Having a vague idea what Span means will not help you solve any exam problems!
Now we have several equivalent ways of making the same statement:

1. A vector $b$ is in the span of $v_1, v_2, \ldots, v_p$.
2. The vector equation

$$x_1 v_1 + x_2 v_2 + \cdots + x_p v_p = b$$

has a solution.
3. The linear system with augmented matrix

$$
\begin{pmatrix}
| & | & \cdots & | & | \\
| v_1 & v_2 & \cdots & v_p & b \\
| & | & \cdots & | & |
\end{pmatrix}
$$

is consistent.

[interactive example]  ←− (this is the picture of an inconsistent linear system)

**Note:** equivalent means that, for any given list of vectors $v_1, v_2, \ldots, v_p, b$, either all three statements are true, or all three statements are false.
Pictures of Span

Drawing a picture of $\text{Span}\{v_1, v_2, \ldots, v_p\}$ is the same as drawing a picture of all linear combinations of $v_1, v_2, \ldots, v_p$.

[interactive: span of two vectors in $\mathbb{R}^2$]
Pictures of Span

In $\mathbb{R}^3$

[interactive: span of two vectors in $\mathbb{R}^3$]  [interactive: span of three vectors in $\mathbb{R}^3$]
Poll

How many vectors are in \( \text{Span}\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \} \)?

A. Zero
B. One
C. Infinity

In general, it appears that \( \text{Span}\{v_1, v_2, \ldots, v_p\} \) is the smallest “linear space” (line, plane, etc.) containing the origin and all of the vectors \( v_1, v_2, \ldots, v_p \).

We will make this precise later.
The whole lecture was about drawing pictures of systems of linear equations.

- **Points** and **vectors** are two ways of drawing elements of $\mathbb{R}^n$. Vectors are drawn as arrows.

- Vector addition, subtraction, and scalar multiplication have geometric interpretations.

- A **linear combination** is a sum of scalar multiples of vectors. This is also a geometric construction, which leads to lots of pretty pictures.

- The **span** of a set of vectors is the set of all linear combinations of those vectors. It is also fun to draw.

- A system of linear equations is equivalent to a vector equation, where the unknowns are the coefficients of a linear combination.