1. For each equation, determine whether the equation is linear or non-linear. Circle your answer. If the equation is non-linear, briefly justify why it is non-linear.
   
   a) \(3x_1 + \sqrt{x_2} = 4\) \hspace{1cm} Linear \hspace{1cm} Not linear
   
   b) \(x_1 = x_2 - x_3 + 10x_4\) \hspace{1cm} Linear \hspace{1cm} Not linear
   
   c) \(e^x + \ln(13)y = \sqrt{2} - z\) \hspace{1cm} Linear \hspace{1cm} Not linear

   **Solution.**
   
   a) Not linear. The \(\sqrt{x_2}\) term makes it non-linear.
   
   b) Linear.
   
   c) Linear. Note \(e^x\) and \(\sqrt{2}\) are just real numbers. Don’t be misled by the appearance of the natural logarithm: \(\ln(13)\) is just the coefficient for \(y\).
   
   If the second term had been \(\ln(13y)\) instead of \(\ln(13)\), then \(y\) would have been inside the logarithm and the equation would have been non-linear.

2. Consider the following three planes, where we use \((x, y, z)\) to denote points in \(\mathbb{R}^3\):

   \[
   \begin{align*}
   2x + 4y + 4z &= 1 \\
   2x + 5y + 2z &= -1 \\
   y + 3z &= 8
   \end{align*}
   \]

   Do all three of the planes intersect? If so, do they intersect at a single point, a line, or a plane?

   **Solution.**

   Subtracting the first equation from the second gives us

   \[
   \begin{align*}
   2x + 4y + 4z &= 1 \\
   y - 2z &= -2 \\
   y + 3z &= 8
   \end{align*}
   \]

   Next, subtracting the second equation from the third gives us

   \[
   \begin{align*}
   2x + 4y + 4z &= 1 \\
   y - 2z &= -2 \\
   5z &= 10, \quad \text{so } z = 2
   \end{align*}
   \]

   We can back-substitute to find \(y\) and then \(x\). The second equation is \(y - 2z = -2\), so \(y - 2(2) = -2\), thus \(y = 2\). The first equation is \(2x + 4(2) + 4(2) = 1\), so \(2x = -15\), thus \(x = -15/2\). We have found that the planes intersect at the point \((\frac{-15}{2}, 2, 2)\).
An alternative method would have been to use augmented matrices to isolate $z$ and then back-substitute:

\[
\begin{pmatrix} 2 & 4 & 4 & 1 \\ 2 & 5 & 2 & -1 \\ 0 & 1 & 3 & 8 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 4 & 4 & 1 \\ 0 & 1 & -2 & -2 \\ 0 & 1 & 3 & 8 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 4 & 4 & 1 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & 5 & 10 \end{pmatrix}
\]

The last line is $5z = 10$, so $z = 2$. From here, back-substitution would give us $y = 2$ and then $x = -\frac{15}{2}$, just like before.

3. Find all values of $h$ so that the lines $x + hy = -5$ and $2x - 8y = 6$ do not intersect. For all such $h$, draw the lines $x + hy = -5$ and $2x - 8y = 6$ to verify that they do not intersect.

**Solution.**

We can use basic algebra, row operations, or geometric intuition.

**Using basic algebra:** Let's see what happens when the lines do intersect. In that case, there is a point $(x, y)$ where

\[
\begin{align*}
x + hy &= -5 \\
2x - 8y &= 6.
\end{align*}
\]

Subtracting twice the first equation from the second equation gives us

\[
x + hy = -5 \\
(-8 - 2h)y = 16.
\]

If $-8 - 2h = 0$ (so $h = -4$), then the second line is $y = 16$, which is impossible. In other words, if $h = -4$ then we cannot find a solution to the system of two equations, so the two lines do not intersect.

On the other hand, if $h \neq -4$, then we can solve for $y$ above:

\[
(-8 - 2h)y = 16 \quad \Rightarrow \quad y = \frac{16}{-8 - 2h} = \frac{8}{-4 - h}.
\]

We can now substitute this value of $y$ into the first equation to find $x$ at the point of intersection:

\[
x + hy = -5 \quad \Rightarrow \quad x + h \cdot \frac{8}{-4 - h} = -5 \quad \Rightarrow \quad x = -5 - \frac{8h}{-4 - h}.
\]

Therefore, the lines fail to intersect if and only if $h = -4$.

**Using intuition from geometry in $\mathbb{R}^2$:** Two non-identical lines in $\mathbb{R}^2$ will fail to intersect, if and only if they are parallel. The second line is $y = \frac{1}{4}x - \frac{3}{4}$, so its slope is $\frac{1}{4}$. If $h \neq 0$, then the first line is $y = -\frac{1}{2}x + \frac{5}{h}$, so the lines are parallel when $-\frac{1}{h} = \frac{1}{4}$, which means $h = -4$. In this case, the lines are $y = \frac{1}{4}x + \frac{5}{4}$ and $y = \frac{1}{4}x - \frac{3}{4}$, so they are parallel non-intersecting lines.

(If $h = 0$ then the first line is vertical and the two lines intersect when $x = -5$).
Using row operations: The problem could be done using augmented matrices, which will soon become our main method for solving systems of equations.

\[
\begin{pmatrix}
1 & h & -5 \\
2 & -8 & 6
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & h & -5 \\
0 & -8-2h & 16
\end{pmatrix}.
\]

If \(-8 - 2h = 0\) (so \(h = -4\)), then the second equation is \(0 = 16\), so our system has no solutions. In other words, the lines do not intersect.

If \(h \neq -4\), then the second equation is \((-8 - 2h)y = 16\), so

\[
y = \frac{16}{-8-2h} = \frac{8}{-4-h} \quad \text{and} \quad x = -5 - hy = -5 - \frac{8h}{-4-h},
\]

and the lines intersect at \((x, y)\). Therefore, our answer is \(h = -4\).

Here are the two lines for \(h = -4\), and we can see they are different parallel lines.

If we vary \(h\) away from \(-4\), then the blue and orange lines will have different slopes and will inevitably intersect. For example,