Math 1553 Worksheet: Fundamentals and §2.1

Solutions

1. For each equation, determine whether the equation is linear or non-linear. Circle your answer. If the equation is non-linear, briefly justify why it is non-linear.

a) $3x_1 + \sqrt{x_2} = 4$ Linear Not linear

b) $x_1 = x_2 - x_3 + 10x_4$ Linear Not linear

c) $e^{\pi}x + \ln(13)y = \sqrt{2} - z$ Linear Not linear

Solution.

a) Not linear. The $\sqrt{x_2}$ term makes it non-linear.

b) Linear.

c) Linear. Note e^{π} and $\sqrt{2}$ are just real numbers. Don't be misled by the appearance of the natural logarithm: $\ln(13)$ is just the coefficient for y.

If the second term had been $\ln(13y)$ instead of $\ln(13)y$, then y would have been inside the logarithm and the equation would have been non-linear.

2. Consider the following three planes, where we use (x, y, z) to denote points in $\mathbf{R}^3$:

$$\begin{aligned} 2x + 4y + 4z &= 1 \\ 2x + 5y + 2z &= -1 \\ y + 3z &= 8 \end{aligned}$$

Do all three of the planes intersect? If so, do they intersect at a single point, a line, or a plane?

Solution.

Subtracting the first equation from the second gives us

$$\begin{aligned} 2x + 4y + 4z &= 1 \\ y - 2z &= -2 \\ y + 3z &= 8. \end{aligned}$$

Next, subtracting the second equation from the third gives us

$$\begin{aligned} 2x + 4y + 4z &= 1 \\ y - 2z &= -2 \\ 5z &= 10, \end{aligned}$$

so $z = 2$. We can back-substitute to find y and then x. The second equation is $y - 2z = -2$, so $y - 2(2) = -2$, thus $y = 2$. The first equation is $2x + 4(2) + 4(2) = 1$, so $2x = -15$, thus $x = -15/2$. We have found that the planes intersect at the point

$$\left(-\frac{15}{2},\ 2,\ 2\right).$$

An alternative method would have been to use augmented matrices to isolate z and then back-substitute:

$$\left(\begin{array}{ccc|c} 2 & 4 & 4 & 1 \\ 2 & 5 & 2 & -1 \\ 0 & 1 & 3 & 8 \end{array}\right) \xrightarrow{R_2=R_2-R_1} \left(\begin{array}{ccc|c} 2 & 4 & 4 & 1 \\ 0 & 1 & -2 & -2 \\ 0 & 1 & 3 & 8 \end{array}\right) \xrightarrow{R_3=R_3-R_2} \left(\begin{array}{ccc|c} 2 & 4 & 4 & 1 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & 5 & 10 \end{array}\right)$$

The last line is $5z = 10$, so $z = 2$. From here, back-substitution would give us $y = 2$ and then $x = -\frac{15}{2}$, just like before.

3. Find all values of h so that the lines $x+hy=-5$ and $2x-8y=6$ do *not* intersect. For all such h, draw the lines $x+hy=-5$ and $2x-8y=6$ to verify that they do not intersect.

Solution.

We can use basic algebra, row operations, or geometric intuition.

Using basic algebra: Let's see what happens when the lines *do* intersect. In that case, there is a point (x, y) where

$$\begin{aligned} x + hy &= -5 \\ 2x - 8y &= 6. \end{aligned}$$

Subtracting twice the first equation from the second equation gives us

$$\begin{aligned} x + \qquad hy &= -5 \\ (-8-2h)y &= 16. \end{aligned}$$

If $-8-2h=0$ (so $h=-4$), then the second line is $0 \cdot y = 16$, which is impossible. In other words, if $h=-4$ then we cannot find a solution to the system of two equations, so the two lines *do not* intersect.

On the other hand, if $h \neq -4$, then we can solve for y above:

$$(-8-2h)y = 16 \qquad y = \frac{16}{-8-2h} \qquad y = \frac{8}{-4-h}.$$

We can now substitute this value of y into the first equation to find x at the point of intersection:

$$x + hy = -5 \qquad x + h \cdot \frac{8}{-4-h} = -5 \qquad x = -5 - \frac{8h}{-4-h}.$$

Therefore, the lines fail to intersect if and only if $\boxed{h=-4}$.

Using intuition from geometry in $\mathbf{R}^2$: Two non-identical lines in $\mathbf{R}^2$ will fail to intersect, if and only if they are parallel. The second line is $y = \frac{1}{4}x - \frac{3}{4}$, so its slope is $\frac{1}{4}$. If $h \neq 0$, then the first line is $y = -\frac{1}{h}x - \frac{5}{h}$, so the lines are parallel when $-\frac{1}{h} = \frac{1}{4}$, which means $h=-4$. In this case, the lines are $y = \frac{1}{4}x + \frac{5}{4}$ and $y = \frac{1}{4}x - \frac{3}{4}$, so they are parallel non-intersecting lines.

(If $h=0$ then the first line is vertical and the two lines intersect when $x=-5$).

Using row operations: The problem could be done using augmented matrices, which will soon become our main method for solving systems of equations.

$$\left(\begin{array}{cc|c} 1 & h & -5 \\ 2 & -8 & 6 \end{array}\right) \xrightarrow{R_2 = R_2 - 2R_1} \left(\begin{array}{cc|c} 1 & h & -5 \\ 0 & -8-2h & 16 \end{array}\right).$$

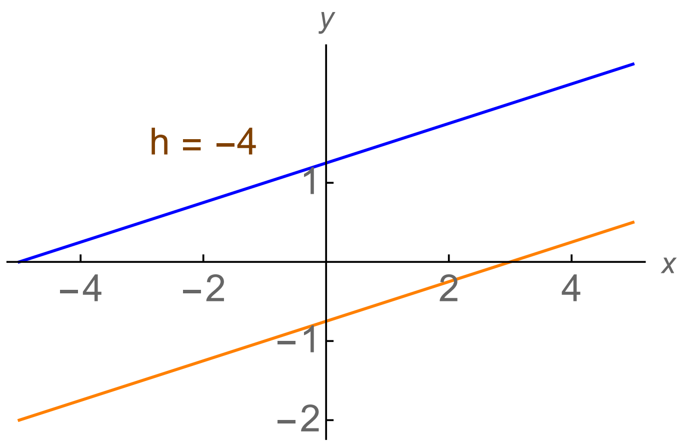
If $-8-2h=0$ (so $h=-4$), then the second equation is $0=16$, so our system has no solutions. In other words, the lines do not intersect.

If $h \neq -4$, then the second equation is $(-8-2h)y=16$, so

$$y = \frac{16}{-8-2h} = \frac{8}{-4-h} \quad \text{and} \quad x = -5 - hy = -5 - \frac{8h}{-4-h},$$

and the lines intersect at (x,y). Therefore, our answer is $h=-4$.

Here are the two lines for $h=-4$, and we can see they are different parallel lines.



If we vary h away from -4, then the blue and orange lines will have different slopes and will inevitably intersect. For example,