MATH 1553
PRACTICE FINAL EXAMINATION

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Please **read all instructions** carefully before beginning.

- The final exam is cumulative, covering all sections and topics on the master calendar.
- Each problem is worth 10 points. The maximum score on this exam is 100 points.
- You have 170 minutes to complete this exam.
- There are no aids of any kind (notes, text, etc.) allowed.
- Please show your work, unless instructed otherwise.
- You may cite any theorem proved in class or in the sections we covered in the text.
- Check your answers if you have time left! Most linear algebra computations can be easily verified for correctness.
- Good luck!

This is a practice exam. It is roughly similar in format, length, and difficulty to the real exam. It is not meant as a comprehensive list of study problems.
Problem 1. [2 points each]

In this problem, you need not explain your answers.

a) The matrix \(\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}\) is in reduced row echelon form:
   1. True   2. False

b) How many solutions does the linear system corresponding to the augmented matrix
   \(\begin{pmatrix} 0 & 1 & 1 & | & 1 \\ 0 & 0 & 0 & | & 1 \end{pmatrix}\) have?
   1. Zero.
   2. One.
   3. Infinity.
   4. Not enough information to determine.

c) Let \(T : \mathbb{R}^n \to \mathbb{R}^m\) be a linear transformation with matrix \(A\). Which of the following
   are equivalent to the statement that \(T\) is one-to-one? (Circle all that apply.)
   1. \(A\) has a pivot in each row.
   2. The columns of \(A\) are linearly independent.
   3. For all vectors \(v, w\) in \(\mathbb{R}^n\), if \(T(v) = T(w)\) then \(v = w\).
   4. \(A\) has \(n\) columns.
   5. \(\text{Nul}A = \{0\}\).

d) Every square matrix has a (real or) complex eigenvalue.
   1. True   2. False

e) Let \(A\) be an \(n \times n\) matrix, and let \(T(x) = Ax\) be the associated matrix transformation. Which of the following
   are equivalent to the statement that \(A\) is not invertible? (Circle all that apply.)
   1. There exists an \(n \times n\) matrix \(B\) such that \(AB = 0\).
   2. \(\text{rank}A = 0\).
   3. \(\det(A) = 0\).
   4. \(\text{Nul}A = \{0\}\).
   5. There exist \(v \neq w\) in \(\mathbb{R}^n\) such that \(T(v) = T(w)\).
Problem 2.  
[2 points each]

In this problem, you need not explain your answers.

a) Let $A$ be an $m \times n$ matrix, and let $b$ be a vector in $\mathbb{R}^m$. Which of the following are equivalent to the statement that $Ax = b$ is consistent? (Circle all that apply.)

1. $b$ is in $\text{Nul} A$.
2. $b$ is in $\text{Col} A$.
3. $A$ has a pivot in every row.
4. The augmented matrix $(A \mid b)$ has no pivot in the last column.

b) Let $A = \begin{pmatrix} 1 & a & 0 \\ 0 & b & 0 \end{pmatrix}$. For what values of $a$ and $b$ is $A$ diagonalizable? (Circle all that apply.)

1. $a = 1, b = 1$
2. $a = 2, b = 1$
3. $a = 1, b = 2$
4. $a = 0, b = 1$

c) Let $W$ be the subset of $\mathbb{R}^2$ consisting of the $x$-axis and the $y$-axis. Which of the following are true? (Circle all that apply.)

1. $W$ contains the zero vector.
2. If $v$ is in $W$, then all scalar multiples of $v$ are in $W$.
3. If $v$ and $w$ are in $W$, then $v + w$ is in $W$.
4. $W$ is a subspace of $\mathbb{R}^2$.

d) Every subspace of $\mathbb{R}^n$ admits an orthogonal basis:

1. True
2. False

e) Let $x$ and $y$ be nonzero orthogonal vectors in $\mathbb{R}^n$. Which of the following are true? (Circle all that apply.)

1. $x \cdot y = 0$
2. $||x - y||^2 = ||x||^2 + ||y||^2$
3. $\text{proj}_{\text{Span}\{x\}}(y) = 0$
4. $\text{proj}_{\text{Span}\{y\}}(x) = 0$
Problem 3. [2 points each]

Short answer questions: you need not explain your answers.

a) Let $A$ be an $n \times n$ matrix. Write the definition of an eigenvector and an eigenvalue of $A$.

b) Suppose $u$ and $v$ are orthogonal unit vectors, and let $x = 2u + v$. Find $\|x\|$.

c) Give an example of a $2 \times 2$ matrix that has no (real) eigenvectors.

d) Let $W$ be the span of $(1, 1, 1, 1)$ in $\mathbb{R}^4$. Find a matrix whose null space is $W^\perp$.

e) Write a $3 \times 3$ matrix $A$ with two (non-real) complex eigenvalues, whose eigenspace corresponding to $\lambda = 7$ is the $x$-axis.
Problem 4. [5 points each]

Let

\[ A = \begin{pmatrix} -5 & 1 & -1 \\ -6 & 5 & 3 \\ 0 & 1 & 1 \end{pmatrix}. \]

a) Compute \( A^{-1} \) and \( \det(A) \).

b) Solve for \( x \) in terms of the variables \( b_1, b_2, b_3 \):

\[ Ax = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}. \]
Problem 5.

Consider the matrix

\[ A = \begin{pmatrix} 2 & 5 & 0 \\ 0 & 1 & 4 \\ 1 & 0 & 5 \end{pmatrix}. \]

a) [4 points] Find an orthogonal basis for ColA.

b) [2 points] Find a different orthogonal basis for ColA. (Reordering and scaling your basis in (a) does not count.)

c) [4 points] Let W be the subspace spanned by \( \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \) and \( \begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix} \). Find the matrix P so that \( Px = \text{proj}_W(x) \) for all \( x \) in \( \mathbb{R}^3 \).
Problem 6.

Suppose that your roommate Jamie is currently taking Math 1551. Jamie scored 72% on the first exam, 81% on the second exam, and 84% on the third exam. Not having taken linear algebra yet, Jamie does not know what kind of score to expect on the final exam. Luckily, you can help out.

a) [4 points] The general equation of a line in $\mathbb{R}^2$ is $y = C + Dx$. Write down the system of linear equations in $C$ and $D$ that would be satisfied by a line passing through the points $(1, 72)$, $(2, 81)$, and $(3, 84)$, and then write down the corresponding matrix equation.

b) [4 points] Solve the corresponding least squares problem for $C$ and $D$, and use this to write down and draw the best fit line below.

![Graph with points and line equation]

\[ y = \square + \square x \]

c) [2 points] What score does this line predict for the fourth (final) exam?
Problem 7.

Consider the vectors

\[ v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}, \quad v_4 = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} \]

and the subspace \( W = \text{Span}\{v_1, v_2, v_3, v_4\} \).

a) [2 points] Find a linear dependence relation among \( v_1, v_2, v_3, v_4 \).

b) [3 points] What is the dimension of \( W \)?

c) [3 points] Which subsets of \( \{v_1, v_2, v_3, v_4\} \) form a basis for \( W \)?

d) [2 points] Choose a basis \( B \) for \( W \) from (c), and find the \( B \)-coordinates of the vector \( w = (0, 0, 4, 0) \).

\[ \text{Hint: it is helpful, but not necessary, to use the fact that } \{v_1, v_2, v_3\} \text{ is orthogonal.} \]
Problem 8.

Let
\[ A = \begin{pmatrix} 1 & 3 & 1 & 1 \\ -1 & -3 & -4 & 2 \\ 5 & 15 & 1 & 9 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 2 \\ 1 \\ 14 \end{pmatrix}. \]

a) [3 points] Find the parametric vector form of the solution set of \( Ax = b \).

b) [2 points] Find a basis for \( \text{Nul} A \).

c) [2 points] What are \( \dim(\text{Nul} A) \) and \( \dim((\text{Nul} A)^\perp) \)?

d) [3 points] Find a basis for \( (\text{Nul} A)^\perp \).
Problem 9.

Consider the matrix

\[ A = \begin{pmatrix} 3 & 2 \\ -10 & 7 \end{pmatrix}. \]

a) [2 points] Compute the characteristic polynomial of \( A \).

b) [2 points] The complex number \( \lambda = 5 - 4i \) is an eigenvalue of \( A \). What is the other eigenvalue? Produce eigenvectors for both eigenvalues.

c) [3 points] Find an invertible matrix \( P \) and a rotation-scaling matrix \( C \) such that \( A = PCP^{-1} \).

d) [1 point] By what factor does \( C \) scale?

e) [2 points] What ray does \( C \) rotate the positive \( x \)-axis onto? Draw it below.
Problem 10.

Let $L$ be a line through the origin in $\mathbb{R}^2$. The reflection over $L$ is the linear transformation $\text{ref}_L : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by

$$\text{ref}_L(x) = x - 2x_{L^\perp} = 2\text{proj}_L(x) - x.$$

a) [3 points] Draw (and label) $\text{ref}_L(u)$, $\text{ref}_L(v)$, and $\text{ref}_L(w)$ in the picture below. [Hint: think geometrically]

In what follows, $L$ does not necessarily refer to the line pictured above.

b) [2 points] If $A$ is the matrix for $\text{ref}_L$, what is $A^2$?

c) [3 points] What are the eigenvalues and eigenspaces of $A$?

d) [2 points] Is $A$ diagonalizable? If so, what diagonal matrix is it similar to?
[Scratch work]
[Scratch work]