Midterm 3

Review Slides
Recall: A set of \( n \) vectors \( \{v_1, v_2, \ldots, v_n\} \) form a basis for \( \mathbb{R}^n \) if and only if the matrix \( C \) with columns \( v_1, v_2, \ldots, v_n \) is invertible.

**Translation:** Let \( B \) be the basis of columns of \( C \). Multiplying by \( C \) changes from the \( B \)-coordinates to the usual coordinates, and multiplying by \( C^{-1} \) changes from the usual coordinates to the \( B \)-coordinates:

\[
[x]_B = C^{-1}x \quad x = C[x]_B.
\]
Similarity

**Definition**
Two \( n \times n \) matrices \( A \) and \( B \) are **similar** if there is an invertible \( n \times n \) matrix \( C \) such that

\[
A = CBC^{-1}.
\]

What does this mean? This gives you a different way of thinking about multiplication by \( A \). Let \( B \) be the basis of columns of \( C \).

To compute \( Ax \), you:
1. multiply \( x \) by \( C^{-1} \) to change to the \( B \)-coordinates: \( [x]_B = C^{-1}x \)
2. multiply this by \( B \): \( B[x]_B = BC^{-1}x \)
3. multiply this by \( C \) to change to usual coordinates: \( Ax = CBC^{-1}x = CB[x]_B \).
**Similarity**

**Definition**
Two $n \times n$ matrices $A$ and $B$ are **similar** if there is an invertible $n \times n$ matrix $C$ such that

$$A = CBC^{-1}.$$ 

What does this mean? This gives you a different way of thinking about multiplication by $A$. Let $B$ be the basis of columns of $C$.

If $A = CBC^{-1}$, then $A$ and $B$ do the same thing, but $B$ operates on the $B$-coordinates, where $B$ is the basis of columns of $C$. 
$A = \begin{pmatrix} \frac{1}{2} & \frac{3}{2} \\ \frac{3}{2} & \frac{1}{2} \end{pmatrix}$  \quad B = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$  \quad C = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$  \quad A = CBC^{-1}$.

What does $B$ do geometrically?

It scales the $x$-direction by 2 and the $y$-direction by $-1$.

To compute $Ax$, first change to the $B$ coordinates, then multiply by $B$, then change back to the usual coordinates, where

$$B = \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} = \{ v_1, v_2 \}$$ (the columns of $C$).
A = \begin{pmatrix} 1/2 & 3/2 \\ 3/2 & 1/2 \end{pmatrix} \quad B = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad A = CBC^{-1}.

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\[ B = \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} = \{v_1, v_2\} \quad \text{(the columns of } C). \]
\( A = \begin{pmatrix} 1/2 & 3/2 \\ 3/2 & 1/2 \end{pmatrix} \quad B = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad A = CBC^{-1}. \)

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To compute \( Ax \), first change to the \( B \) coordinates, then multiply by \( B \), then change back to the usual coordinates, where

\[ B = \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} = \{ v_1, v_2 \} \quad \text{(the columns of } C). \]
$A = \begin{pmatrix} 1/2 & 3/2 \\ 3/2 & 1/2 \end{pmatrix} \quad B = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad A = CBC^{-1}$.

What does $B$ do geometrically?

It scales the $x$-direction by 2 and the $y$-direction by $-1$.

To compute $Ax$, first change to the $B$ coordinates, then multiply by $B$, then change back to the usual coordinates, where

$$B = \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} = \{v_1, v_2\} \quad \text{(the columns of $C$).}$$
What does $A$ do geometrically?

- $B$ scales the $e_1$-direction by 2 and the $e_2$-direction by $-1$.
- $A$ scales the $v_1$-direction by 2 and the $v_2$-direction by $-1$.

Since $B$ is simpler than $A$, this makes it easier to understand $A$.

Note the relationship between the eigenvalues/eigenvectors of $A$ and $B$. 

[interactive]
Similarity
Example (3 × 3)

\[
A = \begin{pmatrix}
-3 & -5 & -3 \\
2 & 4 & 3 \\
-3 & -5 & -2
\end{pmatrix} \quad B = \begin{pmatrix}
2 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 1
\end{pmatrix} \quad C = \begin{pmatrix}
-1 & 1 & 0 \\
1 & -1 & 1 \\
-1 & 0 & 1
\end{pmatrix}
\]

\[\Rightarrow \quad A = CBC^{-1}.\]

What do \(A\) and \(B\) do geometrically?

- \(B\) scales the \(e_1\)-direction by 2, the \(e_2\)-direction by \(-1\), and fixes \(e_3\).
- \(A\) scales the \(v_1\)-direction by 2, the \(v_2\)-direction by \(-1\), and fixes \(v_3\).

Here \(v_1, v_2, v_3\) are the columns of \(C\).
Diagonalizable Matrices

Definition
An \( n \times n \) matrix \( A \) is **diagonalizable** if it is similar to a diagonal matrix:

\[
A = PDP^{-1} \quad \text{for } D \text{ diagonal.}
\]

The Diagonalization Theorem
An \( n \times n \) matrix \( A \) is diagonalizable if and only if \( A \) has \( n \) linearly independent eigenvectors.

In this case, \( A = PDP^{-1} \) for

\[
\begin{align*}
P &= \begin{pmatrix} \mid & \mid & \cdots & \mid \\
v_1 & v_2 & \cdots & v_n \\
\mid & \mid & \cdots & \mid 
\end{pmatrix} \\
D &= \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\
0 & \lambda_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \lambda_n
\end{pmatrix},
\end{align*}
\]

where \( v_1, v_2, \ldots, v_n \) are linearly independent eigenvectors, and \( \lambda_1, \lambda_2, \ldots, \lambda_n \) are the corresponding eigenvalues (in the same order).

Corollary
An \( n \times n \) matrix with \( n \) distinct eigenvalues is diagonalizable.
Definition
Let \( \lambda \) be an eigenvalue of a square matrix \( A \). The **geometric multiplicity** of \( \lambda \) is the dimension of the \( \lambda \)-eigenspace.

Theorem
Let \( \lambda \) be an eigenvalue of a square matrix \( A \). Then
\[
1 \leq \text{(the geometric multiplicity of } \lambda) \leq \text{(the algebraic multiplicity of } \lambda).
\]

Corollary
Let \( \lambda \) be an eigenvalue of a square matrix \( A \). If the algebraic multiplicity of \( \lambda \) is 1, then the geometric multiplicity is also 1.

The Diagonalization Theorem (Alternate Form)
Let \( A \) be an \( n \times n \) matrix. The following are equivalent:
1. \( A \) is diagonalizable.
2. The sum of the geometric multiplicities of the eigenvalues of \( A \) equals \( n \).
3. The sum of the algebraic multiplicities of the eigenvalues of \( A \) equals \( n \), and *the geometric multiplicity equals the algebraic multiplicity* of each eigenvalue.
Algebraic and Geometric Multiplicity

Example

\[
A = \begin{pmatrix}
  7/2 & 0 & 3 \\
-3/2 & 2 & -3 \\
-3/2 & 0 & -1 \\
\end{pmatrix}
\]

Characteristic polynomial:

\[
f(\lambda) = - (\lambda - 2)^2 (\lambda - 1/2)
\]

Algebraic multiplicity of 2: 2
Algebraic multiplicity of 1/2: 1.

Know already:

- The 1/2-eigenspace is a line.
- The 2-eigenspace is a line or a plane.
- The matrix is diagonalizable if and only if the 2-eigenspace is a plane.
Algebraic and Geometric Multiplicity

Example

\[
A - 2I = \begin{pmatrix} 3/2 & 0 & 3 \\ -3/2 & 0 & -3 \\ -3/2 & 0 & -3 \end{pmatrix} \xrightarrow{rref} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]

So a basis for the 2-eigenspace is

\[
\left\{ \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}
\]

This is a \textit{plane}, so the geometric multiplicity is 2.

\[
A - \frac{1}{2}I = \begin{pmatrix} 3 & 0 & 3 \\ -3/2 & 3/2 & -3 \\ -3/2 & 0 & -3/2 \end{pmatrix} \xrightarrow{rref} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}
\]

The \(1/2\)-eigenspace is the \textit{line}

\[
\text{Span} \left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}
\]
Diagonalization

Example

The 2-eigenspace has basis \( \left\{ \begin{pmatrix} -2 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \).

The 1/2-eigenspace has basis \( \left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\} \).

Therefore, \( A = PDP^{-1} \) for

\[
P = \begin{pmatrix} -2 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix} \quad C = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1/2 \end{pmatrix}.
\]

Question: what does \( A \) do geometrically?
Diagonalization
Another example

\[
A = \begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 2
\end{pmatrix}.
\]

The characteristic polynomial is \((x - 1)^2(x - 2)\).

Algebraic multiplicity of 1: 2
Algebraic multiplicity of 2: 1.

Know already:
- The 2-eigenspace is a line.
- The 1-eigenspace is a line or a plane.
- The matrix is diagonalizable if and only if the 1-eigenspace is a plane.

Check: a basis for the 1-eigenspace is \(\{e_1\}\).

Conclusion: \(A\) is not diagonalizable!