The third midterm is on **Friday, November 17**.
- That is one week from this Friday.
- The exam covers §§3.1, 3.2, 5.1, 5.2, 5.3, and 5.5.

WeBWorK 5.1, 5.2 are due today at 11:59pm.

The quiz on Friday covers §§5.1, 5.2.

My office is Skiles 244. Rabin office hours are Monday, 1–3pm and Tuesday, 9–11am.
Section 5.5

Complex Eigenvalues
Consider the matrix for the linear transformation for rotation by $\pi/4$ in the plane. The matrix is:

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$ 

This matrix has no eigenvectors, as you can see geometrically:

or algebraically:

$$f(\lambda) = \lambda^2 - \text{Tr}(A) \lambda + \det(A) = \lambda^2 - \sqrt{2} \lambda + 1 \implies \lambda = \frac{\sqrt{2} \pm \sqrt{-2}}{2}. $$
Complex Numbers

It makes us sad that $-1$ has no square root. If it did, then $\sqrt{-2} = \sqrt{2} \cdot \sqrt{-1}$.

**Mathematician’s solution:** we’re just not using enough numbers! We’re going to declare by *fiat* that there exists a square root of $-1$.

**Definition**
The number $i$ is defined such that $i^2 = -1$.

Once we have $i$, we have to allow numbers like $a + bi$ for real numbers $a, b$.

**Definition**
A *complex number* is a number of the form $a + bi$ for $a, b$ in $\mathbb{R}$. The set of all complex numbers is denoted $\mathbb{C}$.

Note $\mathbb{R}$ is contained in $\mathbb{C}$: they’re the numbers $a + 0i$.

We can identify $\mathbb{C}$ with $\mathbb{R}^2$ by $a + bi \longleftrightarrow (a \ b)$. So when we draw a picture of $\mathbb{C}$, we draw the plane:
In the beginning, people only used counting numbers for, well, counting things: 1, 2, 3, 4, 5, . . . . Then someone (Persian mathematician Muḥammad ibn Mūsā al-Khwārizmī, 825) had the ridiculous idea that there should be a number 0 that represents an absence of quantity. This blew everyone’s mind.

Then it occurred to someone (Chinese mathematician Liu Hui, c. 3rd century) that there should be negative numbers to represent a deficit in quantity. That seemed reasonable, until people realized that 10 − (−3) would have to equal 13. This is when people started saying, “bah, math is just too hard for me.”

At this point it was inconvenient that you couldn’t divide 2 by 3. Thus someone (Indian mathematician Aryabhata, c. 5th century) invented fractions (rational numbers) to represent fractional quantities. These proved very popular. The Pythagoreans developed a whole belief system around the notion that any quantity worth considering could be broken down into whole numbers in this way.

Then the Pythagoreans (c. 6th century BCE) discovered that the hypotenuse of an isosceles right triangle with side length 1 (i.e. $\sqrt{2}$) is not a fraction. This caused a serious existential crisis and led to at least one death by drowning. The real number $\sqrt{2}$ was thus invented to solve the equation $x^2 − 2 = 0$.

So what’s so strange about inventing a number $i$ to solve the equation $x^2 + 1 = 0$? Is this really any stranger than saying an infinite nonrepeating decimal expansion represents a number?
Operations on Complex Numbers

Addition:

Multiplication:

Complex conjugation: \( a + bi = a - bi \) is the complex conjugate of \( a + bi \).

Check: \( \bar{z} + \bar{w} = \bar{z} + \bar{w} \) and \( \bar{z} \bar{w} = \bar{z} \cdot \bar{w} \).

Absolute value: \( |a + bi| = \sqrt{a^2 + b^2} \). This is a real number.

Note: \((a + bi)(a + bi) = (a + bi)(a - bi) = a^2 - (bi)^2 = a^2 + b^2\). So \( |z| = \sqrt{z \bar{z}} \).

Check: \( |zw| = |z| \cdot |w| \).

Division by a nonzero real number: \( \frac{a + bi}{c} = \frac{a}{c} + \frac{b}{c}i \).

Division by a nonzero complex number: \( \frac{z}{w} = \frac{\bar{z} \bar{w}}{ww} = \frac{z \bar{w}}{|w|^2} \).

Example:

\[
\frac{1 + i}{1 - i} =
\]

Real and imaginary part: \( \text{Re}(a + bi) = a \) \quad \text{Im}(a + bi) = b. \)
Polar Coordinates for Complex Numbers

Any complex number $z = a + bi$ has the polar coordinates

$$z = |z|(\cos \theta + i \sin \theta).$$

The angle $\theta$ is called the argument of $z$, and is denoted $\theta = \text{arg}(z)$. Note $\text{arg}(\overline{z}) = -\text{arg}(z)$.

When you multiply complex numbers, you multiply the absolute values and add the arguments:

$$|zw| = |z||w| \quad \text{arg}(zw) = \text{arg}(z) + \text{arg}(w).$$
The Fundamental Theorem of Algebra

The whole point of using complex numbers is to solve polynomial equations. It turns out that they are enough to find all solutions of all polynomial equations:

**Fundamental Theorem of Algebra**

Every polynomial of degree \( n \) has exactly \( n \) complex roots, counted with multiplicity.

Equivalently, if \( f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 \) is a polynomial of degree \( n \), then

\[
f(x) = (x - \lambda_1)(x - \lambda_2) \cdots (x - \lambda_n)
\]

for (not necessarily distinct) complex numbers \( \lambda_1, \lambda_2, \ldots, \lambda_n \).

**Important**

If \( f \) is a polynomial with *real* coefficients, and if \( \lambda \) is a complex root of \( f \), then so is \( \bar{\lambda} \):

\[
0 = \overline{f(\lambda)} = \overline{x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0} = \bar{x}^n + a_{n-1}\bar{x}^{n-1} + \cdots + a_1\bar{x} + a_0 = f(\bar{x}).
\]

Therefore complex roots of real polynomials come in *conjugate pairs*. 

Degree 2: The quadratic formula gives you the (real or complex) roots of any degree-2 polynomial:

\[ f(x) = x^2 + bx + c \implies x = \frac{-b \pm \sqrt{b^2 - 4c}}{2}. \]

For instance, if \( f(\lambda) = \lambda^2 - \sqrt{2}\lambda + 1 \) then

\[ \lambda = \ldots \]

Note the roots are complex conjugates if \( b, c \) are real.
Degree 3: A real cubic polynomial has either three real roots, or one real root and a conjugate pair of complex roots. The graph looks like:

- or

respectively. How do you find a real root? Sometimes you can use this:

**Rational Root Theorem**

Let \( f \) be a polynomial with integer coefficients:

\[
f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0.
\]

Suppose that \( a_0 \neq 0 \) and \( a_n \neq 0 \). If \( \frac{p}{q} \) is a rational root (written in lowest terms), then

- \( p \) divides \( a_0 \), and
- \( q \) divides \( a_n \).
Example: Factor \( f(\lambda) = 5\lambda^3 - 18\lambda^2 + 21\lambda - 10 \).
The characteristic polynomial of

\[ A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \]

is \( f(\lambda) = \lambda^2 - \sqrt{2}\lambda + 1 \). This has two complex roots \( (1 \pm i)/\sqrt{2} \).
Every matrix is guaranteed to have complex eigenvalues and eigenvectors. Using rotation by \( \pi/4 \) from before:

\[
A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad \text{has eigenvalues} \quad \lambda = \frac{1 \pm i}{\sqrt{2}}.
\]

Let’s compute an eigenvector for \( \lambda = (1 + i)/\sqrt{2} \):

\[
A - \lambda I = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1-i \\ 1 & 1-i \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -i & -1 \\ 1 & 0 \end{pmatrix}.
\]

The second row is \( i \) times the first, so we row reduce:

\[
\frac{1}{\sqrt{2}} \begin{pmatrix} -i & -1 \\ 1 & 0 \end{pmatrix} \quad \text{divide by} \quad \frac{-i}{\sqrt{2}} \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.
\]

The parametric form is \( x = i y \), so an eigenvector is \( \begin{pmatrix} i \\ 1 \end{pmatrix} \).

A similar computation shows that an eigenvector for \( \lambda = (1 - i)/\sqrt{2} \) is \( \begin{pmatrix} -i \\ 1 \end{pmatrix} \).

So is \( i \begin{pmatrix} -i \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ i \end{pmatrix} \) (you can scale by complex numbers).
Let $A$ be a $2 \times 2$ matrix, and let $\lambda$ be an eigenvalue of $A$.

Then $A - \lambda I$ is not invertible, so the second row is *automatically* a multiple of the first. (Think about it for a while: otherwise the rref is \( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \).)

Hence the second row disappears in the rref, so we *don’t care what it is*!

If $A - \lambda I = \begin{pmatrix} a & b \\ \cdots & \cdots \end{pmatrix}$, then $(A - \lambda I) \begin{pmatrix} b \\ -a \end{pmatrix} = 0$, so $\begin{pmatrix} b \\ -a \end{pmatrix}$ is an eigenvector.

So is $\begin{pmatrix} -b \\ a \end{pmatrix}$. (What if $a = b = 0$?)

Example:

\[
A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad \lambda = \frac{1 - i}{\sqrt{2}}.
\]
Conjugate Eigenvectors

For \( A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \),

the eigenvalue \( \frac{1+i}{\sqrt{2}} \) has eigenvector \( \begin{pmatrix} i \\ 1 \end{pmatrix} \).

the eigenvalue \( \frac{1-i}{\sqrt{2}} \) has eigenvector \( \begin{pmatrix} -i \\ 1 \end{pmatrix} \).

Do you notice a pattern?

**Fact**

Let \( A \) be a real square matrix. If \( \lambda \) is a complex eigenvalue with eigenvector \( v \), then \( \overline{\lambda} \) is an eigenvalue with eigenvector \( \overline{v} \).

**Why?**

\[
A v = \lambda \implies A \overline{v} = \overline{A v} = \overline{\lambda v} = \overline{\lambda} \overline{v}.
\]

Both eigenvalues and eigenvectors of real square matrices occur in conjugate pairs.
A 3 × 3 Example

Find the eigenvalues and eigenvectors of

\[
A = \begin{pmatrix}
\frac{4}{5} & \frac{-3}{5} & 0 \\
\frac{3}{5} & \frac{4}{5} & 0 \\
0 & 0 & 2
\end{pmatrix}.
\]

The characteristic polynomial is

\[
f(\lambda) = \det \left( \begin{pmatrix}
\frac{4}{5} & \frac{-3}{5} & 0 \\
\frac{3}{5} & \frac{4}{5} & 0 \\
0 & 0 & 2 - \lambda
\end{pmatrix} \right) = (2 - \lambda)(\lambda^2 - \frac{8}{5}\lambda + 1).
\]

This factors out automatically if you expand cofactors along the third row or column.

We computed the roots of this polynomial (times 5) before:

\[
\lambda = 2, \quad \frac{4 + 3i}{5}, \quad \frac{4 - 3i}{5}.
\]

We eyeball an eigenvector with eigenvalue 2 as (0, 0, 1).
A $3 \times 3$ Example
Continued

$$A = \begin{pmatrix}
\frac{4}{5} & -\frac{3}{5} & 0 \\
\frac{3}{5} & \frac{4}{5} & 0 \\
0 & 0 & 2
\end{pmatrix}$$

To find the other eigenvectors, we row reduce:
One can do arithmetic with complex numbers just like real numbers: add, subtract, multiply, divide.

Multiplying complex numbers multiplies the magnitudes and adds the arguments.

An \( n \times n \) matrix always exactly has complex \( n \) eigenvalues, counted with (algebraic) multiplicity.

There’s a trick for computing the (complex) eigenspace of a \( 2 \times 2 \) matrix:

\[
A - \lambda I_2 = \begin{pmatrix} a & b \\ \star & \star \end{pmatrix} \implies \mathbf{v} = \begin{pmatrix} b \\ -a \end{pmatrix} \quad \text{unless } a = b = 0.
\]

The complex eigenvalues and eigenvectors of a real matrix come in complex conjugate pairs:

\[
A \mathbf{v} = \lambda \mathbf{v} \quad \implies \quad A \overline{\mathbf{v}} = \overline{\lambda} \overline{\mathbf{v}}.
\]