▶ WeBWorK 3.1, 3.2 are due today at 11:59pm.

▶ The quiz on Friday covers §§3.1, 3.2.

▶ My office is Skiles 244. Rabin office hours are Monday, 1–3pm and Tuesday, 9–11am.
Section 5.2

The Characteristic Equation
We have a couple of new ways of saying “A is invertible” now:

**The Invertible Matrix Theorem**

Let $A$ be a square $n \times n$ matrix, and let $T : \mathbb{R}^n \to \mathbb{R}^n$ be the linear transformation $T(x) = A x$. The following statements are equivalent.

1. $A$ is invertible.
2. $T$ is invertible.
3. $A$ is row equivalent to $I_n$.
4. $A$ has $n$ pivots.
5. $Ax = 0$ has only the trivial solution.
6. The columns of $A$ are linearly independent.
7. $T$ is one-to-one.
8. $Ax = b$ is consistent for all $b$ in $\mathbb{R}^n$.
9. The columns of $A$ span $\mathbb{R}^n$.
10. $T$ is onto.
11. $A$ has a left inverse (there exists $B$ such that $BA = I_n$).
12. $A$ has a right inverse (there exists $B$ such that $AB = I_n$).
13. $A^T$ is invertible.
14. The columns of $A$ form a basis for $\mathbb{R}^n$.
15. ${\text{Col}} A = \mathbb{R}^n$.
16. $\dim {\text{Col}} A = n$.
17. $\text{rank } A = n$.
18. $\text{Nul } A = \{0\}$.
19. $\dim \text{Nul } A = 0$.
20. The determinant of $A$ is *not* equal to zero.

19. The determinant of $A$ is *not* equal to zero.
20. The number 0 is *not* an eigenvalue of $A$. 
Let $A$ be a square matrix.

$\lambda$ is an eigenvalue of $A \iff Ax = \lambda x$ has a nontrivial solution

$\iff (A - \lambda I)x = 0$ has a nontrivial solution

$\iff A - \lambda I$ is not invertible

$\iff \det(A - \lambda I) = 0$.

This gives us a way to compute the eigenvalues of $A$.

**Definition**

Let $A$ be a square matrix. The **characteristic polynomial** of $A$ is

$$f(\lambda) = \det(A - \lambda I).$$

The **characteristic equation** of $A$ is the equation

$$f(\lambda) = \det(A - \lambda I) = 0.$$

**Important**

The eigenvalues of $A$ are the roots of the characteristic polynomial $f(\lambda) = \det(A - \lambda I)$. 
The Characteristic Polynomial

Example

**Question:** What are the eigenvalues of

\[
A = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}
\]?

**Answer:** First we find the characteristic polynomial:

\[
f(\lambda) = \det(A - \lambda I) = \det \left[ \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right] = \det \begin{pmatrix} 5 - \lambda & 2 \\ 2 & 1 - \lambda \end{pmatrix}
\]

\[
= (5 - \lambda)(1 - \lambda) - 2 \cdot 2
\]

\[
= \lambda^2 - 6\lambda + 1.
\]

The eigenvalues are the roots of the characteristic polynomial, which we can find using the quadratic formula:

\[
\lambda = \frac{6 \pm \sqrt{36 - 4}}{2} = 3 \pm 2\sqrt{2}.
\]
The Characteristic Polynomial

Example

Question: What is the characteristic polynomial of

\[ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \]?

Answer:

\[
f(\lambda) = \det(A - \lambda I) = \det \begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix} = (a - \lambda)(d - \lambda) - bc
\]

\[
= \lambda^2 - (a + d)\lambda + (ad - bc)
\]

What do you notice about \( f(\lambda) \)?

- The constant term is \( \det(A) \), which is zero if and only if \( \lambda = 0 \) is a root.
- The linear term \(- (a + d)\) is the negative of the sum of the diagonal entries of \( A \).

Definition

The **trace** of a square matrix \( A \) is \( \text{Tr}(A) = \text{sum of the diagonal entries of } A \).

Shortcut

The characteristic polynomial of a \( 2 \times 2 \) matrix \( A \) is

\[
f(\lambda) = \lambda^2 - \text{Tr}(A) \lambda + \det(A).
\]
The Characteristic Polynomial

Example

Question: What are the eigenvalues of the rabbit population matrix

\[ A = \begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \]?

Answer: First we find the characteristic polynomial:

\[ f(\lambda) = \det(A - \lambda I) = \det \begin{pmatrix} -\lambda & 6 & 8 \\ \frac{1}{2} & -\lambda & 0 \\ 0 & \frac{1}{2} & -\lambda \end{pmatrix} \]

\[ = 8 \left( \frac{1}{4} - 0 \cdot -\lambda \right) - \lambda \left( \lambda^2 - 6 \cdot \frac{1}{2} \right) \]

\[ = -\lambda^3 + 3\lambda + 2. \]

We know from before that one eigenvalue is \( \lambda = 2 \): indeed, \( f(2) = -8 + 6 + 2 = 0 \). Doing polynomial long division, we get:

\[ \frac{-\lambda^3 + 3\lambda + 2}{\lambda - 2} = -\lambda^2 - 2\lambda - 1 = -(\lambda + 1)^2. \]

Hence \( \lambda = -1 \) is also an eigenvalue.
Definition
The (algebraic) multiplicity of an eigenvalue $\lambda$ is its multiplicity as a root of the characteristic polynomial.

This is not a very interesting notion yet. It will become interesting when we also define geometric multiplicity later.

Example
In the rabbit population matrix, $f(\lambda) = -(\lambda - 2)(\lambda + 1)^2$, so the algebraic multiplicity of the eigenvalue 2 is 1, and the algebraic multiplicity of the eigenvalue $-1$ is 2.

Example
In the matrix \[
\begin{pmatrix}
5 & 2 \\
2 & 1
\end{pmatrix},
\]
$f(\lambda) = (\lambda - (3 - 2\sqrt{2}))(\lambda - (3 + 2\sqrt{2}))$, so the algebraic multiplicity of $3 + 2\sqrt{2}$ is 1, and the algebraic multiplicity of $3 - 2\sqrt{2}$ is 1.
Fact: If $A$ is an $n \times n$ matrix, the characteristic polynomial

$$f(\lambda) = \det(A - \lambda I)$$

turns out to be a polynomial of degree $n$, and its roots are the eigenvalues of $A$:

$$f(\lambda) = (-1)^n \lambda^n + a_{n-1} \lambda^{n-1} + a_{n-2} \lambda^{n-2} + \cdots + a_1 \lambda + a_0.$$ 

Poll

If you count the eigenvalues of $A$, with their algebraic multiplicities, you will get:

A. Always $n$.
B. Always at most $n$, but sometimes less.
C. Always at least $n$, but sometimes more.
D. None of the above.

The answer depends on whether you allow complex eigenvalues. If you only allow real eigenvalues, the answer is B. Otherwise it is A, because any degree-$n$ polynomial has exactly $n$ complex roots, counted with multiplicity. Stay tuned.
The $B$-basis
Review

Recall: If \{v_1, v_2, \ldots, v_m\} is a basis for a subspace $V$ and $x$ is in $V$, then the $B$-coordinates of $x$ are the (unique) coefficients $c_1, c_2, \ldots, c_m$ such that

$$x = c_1 v_1 + c_2 v_2 + \cdots + c_m v_m.$$ 

In this case, the $B$-coordinate vector of $x$ is

$$[x]_B = \begin{pmatrix}
c_1 \\
c_2 \\
\vdots \\
c_m
\end{pmatrix}.$$ 

Example: The vectors

$$v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

form a basis for $\mathbb{R}^2$ because they are not collinear. [interactive]
Coordinate Systems on $\mathbb{R}^n$

**Recall:** A set of $n$ vectors $\{v_1, v_2, \ldots, v_n\}$ form a basis for $\mathbb{R}^n$ if and only if the matrix $C$ with columns $v_1, v_2, \ldots, v_n$ is invertible.

If $x = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$ then

$$[x]_B = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \implies x = c_1 v_1 + c_2 v_2 + c_n v_n = C[x]_B.$$  

Since $x = C[x]_B$ we have $[x]_B = C^{-1} x$.

**Translation:** Let $B$ be the basis of columns of $C$. Multiplying by $C$ changes from the $B$-coordinates to the usual coordinates, and multiplying by $C^{-1}$ changes from the usual coordinates to the $B$-coordinates:

$$[x]_B = C^{-1} x \quad \text{and} \quad x = C[x]_B.$$
**Definition**

Two $n \times n$ matrices $A$ and $B$ are **similar** if there is an invertible $n \times n$ matrix $C$ such that

$$A = CBC^{-1}.$$ 

What does this mean? This gives you a different way of thinking about multiplication by $A$. Let $B$ be the basis of columns of $C$.

To compute $Ax$, you:

1. multiply $x$ by $C^{-1}$ to change to the $B$-coordinates: $[x]_B = C^{-1}x$
2. multiply this by $B$: $B[x]_B = BC^{-1}x$
3. multiply this by $C$ to change to usual coordinates: $Ax = CBC^{-1}x = CB[x]_B$. 
**Similarity**

**Definition**
Two $n \times n$ matrices $A$ and $B$ are **similar** if there is an invertible $n \times n$ matrix $C$ such that

$$A = CBC^{-1}.$$ 

What does this mean? This gives you a different way of thinking about multiplication by $A$. Let $B$ be the basis of columns of $C$.

If $A = CBC^{-1}$, then $A$ and $B$ do the same thing, but $B$ operates on the $B$-coordinates, where $B$ is the basis of columns of $C$. 
$A = \begin{pmatrix} 1/2 & 3/2 \\ 3/2 & 1/2 \end{pmatrix}$  \hspace{1cm}  $B = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$ \hspace{1cm}  $C = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$  \hspace{1cm}  $A = CBC^{-1}$.

What does $B$ do geometrically?

It scales the $x$-direction by 2 and the $y$-direction by $-1$.

To compute $Ax$, first change to the $B$ coordinates, then multiply by $B$, then change back to the usual coordinates, where

$$B = \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} = \{ v_1, v_2 \} \quad \text{(the columns of $C$)}.$$
$A = \begin{pmatrix} 1/2 & 3/2 \\ 3/2 & 1/2 \end{pmatrix}$  

$B = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$  

$C = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$  

$A = CBC^{-1}$.

What does $B$ do geometrically?

It scales the $x$-direction by 2 and the $y$-direction by $-1$.

To compute $Ax$, first change to the $B$ coordinates, then multiply by $B$, then change back to the usual coordinates, where

$B = \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} = \{v_1, v_2\}$ (the columns of $C$).
Similarity

Example

\[
A = \begin{pmatrix}
\frac{1}{2} & \frac{3}{2} \\
\frac{3}{2} & \frac{1}{2}
\end{pmatrix} \quad B = \begin{pmatrix}
2 & 0 \\
0 & -1
\end{pmatrix} \quad C = \begin{pmatrix}
1 & 1 \\
1 & -1
\end{pmatrix} \quad A = CBC^{-1}.
\]

What does \(B\) do geometrically?

It scales the \(x\)-direction by 2 and the \(y\)-direction by \(-1\).

To compute \(Ax\), first change to the \(B\) coordinates, then multiply by \(B\), then change back to the usual coordinates, where

\[
B = \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} = \{v_1, v_2\} \quad \text{(the columns of \(C\)).}
\]
What does $B$ do geometrically?

It scales the $x$-direction by 2 and the $y$-direction by $-1$.

To compute $Ax$, first change to the $B$ coordinates, then multiply by $B$, then change back to the usual coordinates, where

$$B = \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} = \{v_1, v_2\} \quad \text{(the columns of $C$).}$$
What does $A$ do geometrically?

- $B$ scales the $e_1$-direction by 2 and the $e_2$-direction by $-1$.
- $A$ scales the $v_1$-direction by 2 and the $v_2$-direction by $-1$.

Since $B$ is simpler than $A$, this makes it easier to understand $A$.
Note the relationship between the eigenvalues/eigenvectors of $A$ and $B$. 
Similarity
Example (3 × 3)

\[
A = \begin{pmatrix}
-3 & -5 & -3 \\
2 & 4 & 3 \\
-3 & -5 & -2
\end{pmatrix}
B = \begin{pmatrix}
2 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 1
\end{pmatrix}
C = \begin{pmatrix}
-1 & 1 & 0 \\
1 & -1 & 1 \\
-1 & 0 & 1
\end{pmatrix}
\]

\[
A = CBC^{-1}.
\]

What do \(A\) and \(B\) do geometrically?

- \(B\) scales the \(e_1\)-direction by 2, the \(e_2\)-direction by \(-1\), and fixes \(e_3\).
- \(A\) scales the \(v_1\)-direction by 2, the \(v_2\)-direction by \(-1\), and fixes \(v_3\).

Here \(v_1, v_2, v_3\) are the columns of \(C\).
Fact: If $A$ and $B$ are similar, then they have the same characteristic polynomial.

Why? Suppose $A = CBC^{-1}$.

\[
A - \lambda I = CBC^{-1} - \lambda I
\]
\[
= CBC^{-1} - C(\lambda I)C^{-1}
\]
\[
= C(B - \lambda I)C^{-1}.
\]

Therefore,

\[
\det(A - \lambda I) = \det(C(B - \lambda I)C^{-1})
\]
\[
= \det(C) \det(B - \lambda I) \det(C^{-1})
\]
\[
= \det(B - \lambda I),
\]

because $\det(C^{-1}) = \det(C)^{-1}$.

Consequence: similar matrices have the same eigenvalues! (But different eigenvectors in general.)
1. Matrices with the same eigenvalues need not be similar. For instance,

\[
\begin{pmatrix}
2 & 1 \\
0 & 2
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
2 & 0 \\
0 & 2
\end{pmatrix}
\]

both only have the eigenvalue 2, but they are not similar.

2. Similarity has nothing to do with row equivalence. For instance,

\[
\begin{pmatrix}
2 & 1 \\
0 & 2
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\]

are row equivalent, but they have different eigenvalues.
We did two different things today.

First we talked about characteristic polynomials:

- We learned to find the eigenvalues of a matrix by computing the roots of the characteristic polynomial $p(\lambda) = \det(A - \lambda I)$.
- For a $2 \times 2$ matrix $A$, the characteristic polynomial is just
  $$p(\lambda) = \lambda^2 - \text{Tr}(A)\lambda + \det(A).$$
- The **algebraic multiplicity** of an eigenvalue is its multiplicity as a root of the characteristic polynomial.

Then we talked about similar matrices:

- Two square matrices $A, B$ of the same size are **similar** if there is an invertible matrix $C$ such that $A = CBC^{-1}$.
- Geometrically, similar matrices $A$ and $B$ do the same thing, except $B$ operates on the coordinate system $\mathcal{B}$ defined by the columns of $C$:
  $$B[x]_\mathcal{B} = [Ax]_\mathcal{B}.$$
- This is useful when we can find a similar matrix $B$ which is **simpler** than $A$ (e.g., a diagonal matrix).