Announcements

Wednesday, October 18

▶ The second midterm is on **this Friday, October 20**.
  ▶ The exam covers §§ 1.7, 1.8, 1.9, 2.1, 2.2, 2.3, 2.8, and 2.9.
  ▶ About half the problems will be conceptual, and the other half computational.
  ▶ Note that this midterm covers more material than the first!

▶ There is a practice midterm posted on the website. It is identical in format to the real midterm (although there may be ±1–2 problems).

▶ Study tips:
  ▶ There are lots of problems at the end of each section in the book, and at the end of the chapter, for practice.
  ▶ Make sure to learn the theorems and learn the definitions, and understand what they mean. There is a reference sheet on the website.
  ▶ Sit down to do the practice midterm in 50 minutes, with no notes.
  ▶ Come to office hours!

▶ WeBWorK 2.8, 2.9 are due today at 11:59pm.

▶ **Double Rabin office hours this week**: Monday, 1–3pm; Tuesday, 9–11am; Thursday, 9–11am; Thursday, 12–2pm.

▶ **TA review session**: Today, 7:15–9pm, Culc 144.
Midterm 2

Review Slides
Definition

A transformation (or function or map) from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule $T$ that assigns to each vector $x$ in $\mathbb{R}^n$ a vector $T(x)$ in $\mathbb{R}^m$.

- $\mathbb{R}^n$ is called the domain of $T$ (the inputs).
- $\mathbb{R}^m$ is called the codomain of $T$ (the outputs).
- For $x$ in $\mathbb{R}^n$, the vector $T(x)$ in $\mathbb{R}^m$ is the image of $x$ under $T$.
  Notation: $x \mapsto T(x)$.
- The set of all images $\{T(x) \mid x \text{ in } \mathbb{R}^n\}$ is the range of $T$.

Notation:

$$T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$ means $T$ is a transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$.

It may help to think of $T$ as a “machine” that takes $x$ as an input, and gives you $T(x)$ as the output.
Matrix Transformations

If $A$ is an $m \times n$ matrix, then

$$T : \mathbb{R}^n \to \mathbb{R}^m \text{ defined by } T(x) = Ax$$

is a matrix transformation.

These are the kinds of transformations we can use linear algebra to study, because they come from matrices.

Example: $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$

$$T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + 2y + 3z \\ 4x + 5y + 6z \end{pmatrix}$$

(Note we've written a formula for $T$ that doesn't a priori have anything to do with matrices.)
Questions about Transformations

Here are some natural questions that one can ask about a general transformation (not just on the midterm, but in the real world too):

**Question:** What kind of vectors can you input into $T$? What kind of vectors do you get out? In other words, what are the domain and codomain?

**Answer for $T(x) = Ax$:** Inputs are in $\mathbb{R}^n$, where $n$ is the number of *columns* of $T$. Outputs are in $\mathbb{R}^m$, where $m$ is the number of *rows* of $A$. (Cf. previous slide.)

**Question:** For which $b$ does $T(x) = b$ have a solution? In other words, what is the range of $T$?

**Answer for $T(x) = Ax$:** The range is $\text{Col } A$, the span of the columns: $T(x) = Ax$ is a linear combination of the columns of $A$.

**Question:** Is $T$ one-to-one, onto, and/or invertible?

**Answer for $T(x) = Ax$:** on the next slides
One-to-one and onto

**Definition**
A transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is:

- **one-to-one** if $T(x) = b$ has at most one solution for every $b$ in $\mathbb{R}^m$  
- **onto** if $T(x) = b$ has at least one solution for every $b$ in $\mathbb{R}^m$

**Picture:**  [interactive]

This is neither one-to-one nor onto.

- Can you find two different solutions to $T(x) = 0$?
- Can you find a $b$ such that $T(x) = b$ has no solution?

**Picture:**  [interactive]

This is onto but not one-to-one.

- Can you find two different solutions to $T(x) = 0$?

**Picture:**  [interactive]

This is one-to-one and onto.
Theorem
Let \( T : \mathbb{R}^n \to \mathbb{R}^m \) be a matrix transformation with matrix \( A \). Then the following are equivalent:

- \( T \) is one-to-one
- \( T(x) = b \) has one or zero solutions for every \( b \) in \( \mathbb{R}^m \)
- \( Ax = b \) has a unique solution or is inconsistent for every \( b \) in \( \mathbb{R}^m \)
- \( Ax = 0 \) has a unique solution
- The columns of \( A \) are linearly independent
- \( A \) has a pivot in column.

Theorem
Let \( T : \mathbb{R}^n \to \mathbb{R}^m \) be a matrix transformation with matrix \( A \). Then the following are equivalent:

- \( T \) is onto
- \( T(x) = b \) has a solution for every \( b \) in \( \mathbb{R}^m \)
- \( Ax = b \) is consistent for every \( b \) in \( \mathbb{R}^m \)
- The columns of \( A \) span \( \mathbb{R}^m \)
- \( A \) has a pivot in every row
Question: How do you know if a transformation is a matrix transformation or not?

Definition
A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear if it satisfies the equations

$$T(u + v) = T(u) + T(v) \quad \text{and} \quad T(cv) = cT(v).$$

for all vectors $u, v$ in $\mathbb{R}^n$ and all scalars $c$. ($\implies T(0) = 0$)

Theorem
Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then $T$ is a matrix transformation with matrix

$$A = \begin{pmatrix} T(e_1) & T(e_2) & \cdots & T(e_n) \end{pmatrix}.$$

So a linear transformation is a matrix transformation, where you haven’t computed the matrix yet.

Important
You compute the columns of the matrix for $T$ by plugging in $e_1, e_2, \ldots, e_n$. 

Example: $T: \mathbb{R} \rightarrow \mathbb{R}$ defined by $T(x) = x + 1$.

This is not linear: $T(0) = 1 \neq 0$.

Example: $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by rotation by $\theta$ degrees. Is $T$ linear? Check:

The pictures show $T(u) + T(v) = T(u + v)$ and $T(cu) = cT(u)$, so $T$ is linear.
Example: \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) defined by rotation by \( \theta \) degrees. What is the standard matrix?

\[
T(e_1) = \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix}
\quad \text{and} \quad
T(e_2) = \begin{pmatrix} -\sin(\theta) \\ \cos(\theta) \end{pmatrix}
\]

\[
\Longrightarrow A = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}
\]
Example: \( T : \mathbb{R}^3 \rightarrow \mathbb{R}^2 \) defined by
\[
T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x + 3y - z \\ y + z \end{pmatrix}.
\]

Is \( T \) linear? Check \( T(u + v) = T(u) + T(v) \):
\[
T \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + T \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} 2x_1 + 3y_1 - z_1 \\ y_1 + z_1 \end{pmatrix} + \begin{pmatrix} 2x_2 + 3y_2 - z_2 \\ y_2 + z_2 \end{pmatrix}
\]

These are equal. \( \checkmark \)

Note we’re treating \( u \) and \( v \) as \textit{unknown} vectors: this has to work for all vectors \( u \) and \( v \)!
Example: $T: \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x + 3y - z \\ y + z \end{pmatrix}.$$ 

Is $T$ linear? Check $T(cu) = cT(u)$:

$$T \begin{pmatrix} c \\ x \\ y \\ z \end{pmatrix} = T \begin{pmatrix} cx \\ cy \\ cz \end{pmatrix} = \begin{pmatrix} 2cx + 3cy - cz \\ cy + cz \end{pmatrix}$$

$$cT \begin{pmatrix} x \\ y \\ z \end{pmatrix} = c \begin{pmatrix} 2x + 3y - z \\ y + z \end{pmatrix} = \begin{pmatrix} c(2x + 3y - z) \\ c(y + z) \end{pmatrix}$$

These are equal.  

Conclusion: $T$ is linear.
Example: \( T : \mathbb{R}^3 \rightarrow \mathbb{R}^2 \) defined by

\[
T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x + 3y - z \\ y + z \end{pmatrix}.
\]

We know it is linear, so it is a matrix transformation. What is its standard matrix \( A \)?

\[
T(e_1) = T \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}
\]

\[
T(e_2) = T \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad \Rightarrow \quad A = \begin{pmatrix} 2 & 3 & -1 \\ 0 & 1 & 1 \end{pmatrix}.
\]

\[
T(e_3) = T \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}
\]
Definition
A **subspace** of \( \mathbb{R}^n \) is a subset \( V \) of \( \mathbb{R}^n \) satisfying:

1. The zero vector is in \( V \). “not empty”
2. If \( u \) and \( v \) are in \( V \), then \( u + v \) is also in \( V \). “closed under addition”
3. If \( u \) is in \( V \) and \( c \) is in \( \mathbb{R} \), then \( cu \) is in \( V \). “closed under \( \times \) scalars”

A subspace is a span, and a span is a subspace.

Important examples of subspaces:
- The span of any set of vectors.
- The column space of a matrix.
- The null space of a matrix.
- The solution set of a system of homogeneous equations.
- All of \( \mathbb{R}^n \) and the zero subspace \( \{0\} \).
Subspaces
What is the point?

The point of a subspace is to talk about a span without figuring out which vectors it’s the span of.

Example: 
\[ A = \begin{pmatrix} 2 & 7 & -4 & 3 \\ 0 & 0 & 12 & 1 \\ 0 & 0 & 0 & -78 \end{pmatrix} \]

\[ V = \text{Nul} \ A \]

There are 3 pivots, so rank \( A = 3 \).

By the rank theorem, \( \dim \text{Nul} \ A = 1 \).

We know the null space is a line, but we never had to compute a spanning vector!