Math 1553 Worksheet §2.8

1. Find bases for the column space and the null space of

\[ A = \begin{pmatrix} 0 & 1 & -3 & 1 & 0 \\ 1 & -1 & 8 & -7 & 1 \\ -1 & -2 & 1 & 4 & -1 \end{pmatrix}. \]

2. Consider the following vectors in \( \mathbb{R}^3 \):

\[ b_1 = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}, \quad b_2 = \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix}, \quad u = \begin{pmatrix} 1 \\ 10 \\ 7 \end{pmatrix} \]

Let \( V = \text{Span}\{b_1, b_2\} \).

a) Explain why \( B = \{b_1, b_2\} \) is a basis for \( V \).

b) Determine if \( u \) is in \( V \).

c) Find a vector \( b_3 \) such that \( \{b_1, b_2, b_3\} \) is a basis of \( \mathbb{R}^3 \).

3. For (a) and (b), answer “yes” if the statement is always true, “no” if it is always false, and “maybe” otherwise.

a) If \( A \) is an \( n \times n \) matrix and \( \text{Col} \ A = \mathbb{R}^n \), then \( Ax = 0 \) has a nontrivial solution.

b) If \( A \) is an \( m \times n \) matrix and \( Ax = 0 \) has only the trivial solution, then the columns of \( A \) form a basis for \( \mathbb{R}^m \).

c) Give an example of a \( 2 \times 2 \) matrix whose column space is the same as its null space.

4. In each case, determine whether the given set is a subspace of \( \mathbb{R}^4 \). If it is a subspace, justify why. If it is not a subspace, state a subspace property that it fails.

a) \( V = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \in \mathbb{R}^4 \mid x + y = 0 \text{ and } z + w = 0 \right\} \)

b) \( W = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \in \mathbb{R}^4 \mid xy - zw = 0 \right\} \)

5. This problem covers section 2.9. Parts (a), (b), and (c) are unrelated to each other.

a) True or false: If \( A \) is a \( 3 \times 100 \) matrix of rank 2, then \( \text{dim}(\text{Nul} A) = 97 \).

b) For \( u \) and \( B \) from problem 2, find \([u]_B\) (the \( B \)-coordinates of \( u \)).

c) Let \( D = \left\{ \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \end{pmatrix} \right\} \), and suppose \([x]_D = \begin{pmatrix} -1 \\ 3 \end{pmatrix} \). Find \( x \).