Math 1553 Worksheet §2.8

1. Find bases for the column space and the null space of

\[
A = \begin{pmatrix}
0 & 1 & -3 & 1 & 0 \\
1 & -1 & 8 & -7 & 1 \\
-1 & -2 & 1 & 4 & -1
\end{pmatrix}.
\]

**Solution.**

Finding a basis for \( \text{Nul} \ A \) means finding the parametric vector form of the solution to \( Ax = 0 \). First we row reduce:

\[
\begin{pmatrix}
0 & 1 & -3 & 1 & 0 \\
1 & -1 & 8 & -7 & 1 \\
-1 & -2 & 1 & 4 & -1
\end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix}
1 & 0 & 5 & -6 & 1 \\
0 & 1 & -3 & 1 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix},
\]

so \( x_3, x_4, x_5 \) are free, and

\[
\begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5
\end{pmatrix} = \begin{pmatrix}
-5x_3 + 6x_4 - x_5 \\
3x_3 - x_4 \\
x_3 \\
x_4 \\
x_5
\end{pmatrix} = x_3 \begin{pmatrix}
-5 \\
3 \\
1 \\
0 \\
0
\end{pmatrix} + x_4 \begin{pmatrix}
6 \\
-1 \\
0 \\
1 \\
0
\end{pmatrix} + x_5 \begin{pmatrix}
-1 \\
0 \\
0 \\
0 \\
1
\end{pmatrix}.
\]

Therefore, a basis for \( \text{Nul} \ A \) is

\[
\left\{ \begin{pmatrix}
-5 \\
3 \\
1 \\
0 \\
0
\end{pmatrix}, \begin{pmatrix}
6 \\
-1 \\
0 \\
1 \\
0
\end{pmatrix}, \begin{pmatrix}
-1 \\
0 \\
0 \\
0 \\
1
\end{pmatrix} \right\}.
\]

To find a basis for \( \text{Col} \ A \), we use the pivot columns as they were written in the original matrix \( A \), not its RREF. These are the first two columns:

\[
\left\{ \begin{pmatrix}
0 \\
1 \\
-1 \\
-2
\end{pmatrix}, \begin{pmatrix}
1 \\
4 \\
3 \\
7
\end{pmatrix} \right\}.
\]

2. Consider the following vectors in \( \mathbb{R}^3 \):

\[
b_1 = \begin{pmatrix}
2 \\
2 \\
2
\end{pmatrix} \quad b_2 = \begin{pmatrix}
1 \\
4 \\
3
\end{pmatrix} \quad u = \begin{pmatrix}
1 \\
10 \\
7
\end{pmatrix}
\]

Let \( V = \text{Span}\{b_1, b_2\} \).

a) Explain why \( B = \{b_1, b_2\} \) is a basis for \( V \).

b) Determine if \( u \) is in \( V \).

c) Find a vector \( b_3 \) such that \( \{b_1, b_2, b_3\} \) is a basis of \( \mathbb{R}^3 \).

**Solution.**
a) A quick check shows that \( b_1 \) and \( b_2 \) are linearly independent (verify!), and we already know they span \( V \), so \( \{b_1, b_2\} \) is a basis for \( V \).

b) \( u \) is in \( V \) if and only if \( c_1 b_1 + c_2 b_2 = u \) for some \( c_1 \) and \( c_2 \) (in which case \([u]_B = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}\) looking ahead to problem 5(b)). We form the augmented matrix \( \begin{pmatrix} b_1 & b_2 & u \end{pmatrix} \) and see if the system is consistent.

\[
\begin{array}{ccc|c}
2 & 1 & 1 \\
2 & 4 & 10 \\
2 & 3 & 7 \\
\end{array} \rightarrow 
\begin{array}{ccc|c}
2 & 1 & 1 \\
0 & 3 & 9 \\
0 & 2 & 6 \\
\end{array} \rightarrow 
\begin{array}{ccc|c}
2 & 1 & 1 \\
0 & 1 & 3 \\
0 & 0 & 0 \\
\end{array}.
\]

The right column is not a pivot column, so the system is consistent, therefore \( u \) is in \( \text{Span}\{b_1, b_2\} \): in fact, \( u = -b_1 + 3b_2 \).

c) If we choose \( b_3 \) which is not in \( \text{Span}\{b_1, b_2\} \), then \( \{b_1, b_2, b_3\} \) is linearly independent by the increasing span criterion. Any three linearly independent vectors span \( \mathbb{R}^3 \): the matrix with columns \( b_1, b_2, b_3 \) is square, so if there is a pivot in every column, then there is a pivot in every row.

We could choose \( b_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \), since \( \begin{pmatrix} b_1 & b_2 & b_3 \end{pmatrix} \) is inconsistent:

\[
\begin{array}{ccc|c}
2 & 1 & 1 \\
2 & 4 & 0 \\
2 & 3 & 0 \\
\end{array} \rightarrow 
\begin{array}{ccc|c}
2 & 1 & 0 \\
0 & 3 & -1 \\
0 & 2 & -1 \\
\end{array} \rightarrow 
\begin{array}{ccc|c}
2 & 1 & 0 \\
0 & 1 & -1 \\
0 & 0 & -1/3 \\
\end{array}.
\]

3. For (a) and (b), answer “yes” if the statement is always true, “no” if it is always false, and “maybe” otherwise.

a) If \( A \) is an \( n \times n \) matrix and \( \text{Col } A = \mathbb{R}^n \), then \( Ax = 0 \) has a nontrivial solution.

b) If \( A \) is an \( m \times n \) matrix and \( Ax = 0 \) has only the trivial solution, then the columns of \( A \) form a basis for \( \mathbb{R}^m \).

c) Give an example of \( 2 \times 2 \) matrix whose column space is the same as its null space.

**Solution.**

a) No. Since \( \text{Col}(A) = \mathbb{R}^n \), the linear transformation \( T(x) = Ax \) from \( \mathbb{R}^n \) to \( \mathbb{R}^n \) is onto, hence \( T \) is one-to-one, so \( Ax = 0 \) has only the trivial solution.

b) Maybe. If \( Ax = 0 \) has only the trivial solution and \( m = n \), then \( A \) is invertible, so the columns of \( A \) are linearly independent and span \( \mathbb{R}^m \).

If \( m > n \) then the statement is false. For example, \( A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \) has only the trivial solution for \( Ax = 0 \), but its columns form only a 2-plane within \( \mathbb{R}^3 \).

c) Take \( A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \). Its null space and column space are \( \text{Span}\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\} \).
4. In each case, determine whether the given set is a subspace of \( \mathbb{R}^4 \). If it is a subspace, justify why. If it is not a subspace, state a subspace property that it fails.

\[ a) \ V = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \in \mathbb{R}^4 \mid x + y = 0 \text{ and } z + w = 0 \right\} \]

\[ b) \ W = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \in \mathbb{R}^4 \mid xy - zw = 0 \right\} \]

Solution.

\[ a) \text{ The condition } "x + y = 0 \text{ and } z + w = 0" \text{ means that the vectors in } V \text{ are the solutions to the system of homogeneous equations} \]

\[
\begin{align*}
x + y &= 0 \\
z + w &= 0.
\end{align*}
\]

In other words, \( V \) is the null space of the matrix

\[
\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.
\]

A null space is automatically a subspace, so \( V \) is a subspace.

Alternatively, we can verify the subspace properties:

(1) The zero vector is in \( V \), since \( 0 + 0 = 0 \) and \( 0 + 0 = 0 \).

(2) If \( u = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \\ w_1 \end{pmatrix} \) and \( v = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \\ w_2 \end{pmatrix} \) are in \( V \). Compute \( u + v = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \\ w_1 + w_2 \end{pmatrix} \).

Are \( (x_1 + x_2) + (y_1 + y_2) = 0 \) and \( (z_1 + z_2) + (w_1 + w_2) = 0 \)? Yes:

\[
\begin{align*}
(x_1 + x_2) + (y_1 + y_2) &= (x_1 + y_1) + (x_2 + y_2) = 0 + 0 = 0, \\
(z_1 + z_2) + (w_1 + w_2) &= (z_1 + w_1) + (z_2 + w_2) = 0 + 0 = 0.
\end{align*}
\]

(3) If \( u = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \\ w_1 \end{pmatrix} \) is in \( V \) then so is \( cu \) for any scalar:

\[
cx_1 + cy_1 = c(x_1 + y_1) = c(0) = 0, \quad cz_1 + cw_1 = c(z_1 + w_1) = c(0) = 0.
\]
b) Not a subspace. Note \( u = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \) and \( v = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \) are in \( W \), but \( u + v \) is not in \( W \).

\[
u + v = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad 1 \cdot 1 - 0 \cdot 0 = 1 \neq 0. \quad (W \text{ is not closed under addition})
\]

5. This problem covers section 2.9. Parts (a), (b), and (c) are unrelated to each other.

a) True or false: If \( A \) is a \( 3 \times 100 \) matrix of rank 2, then \( \dim(Nul A) = 97 \).

b) For \( u \) and \( \mathcal{B} \) from problem 2, find \([u]_{\mathcal{B}}\) (the \( \mathcal{B} \)-coordinates of \( u \)).

c) Let \( D = \left\{ \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \end{pmatrix} \right\} \), and suppose \([x]_{D} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}\). Find \( x \).

**Solution.**

a) No. By the Rank Theorem, \( \text{rank}(A) + \dim(Nul A) = 100 \), so \( \dim(Nul A) = 98 \).

b) \( u \) is in \( V \) if and only if \( c_1 b_1 + c_2 b_2 = u \) for some \( c_1 \) and \( c_2 \), in which case \([u]_{\mathcal{B}} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}\). We form the augmented matrix \( \begin{pmatrix} b_1 & b_2 & u \end{pmatrix} \) and solve:

\[
\begin{pmatrix}
2 & 1 & 1 \\
2 & 4 & 10 \\
2 & 3 & 7
\end{pmatrix}
\xrightarrow{R_2 = R_2 - R_1}
\begin{pmatrix}
2 & 1 & 1 \\
0 & 3 & 9 \\
0 & 2 & 6
\end{pmatrix}
\xrightarrow{R_3 = R_3 - R_1}
\begin{pmatrix}
2 & 1 & 1 \\
0 & 3 & 6 \\
0 & 0 & 0
\end{pmatrix}
\xrightarrow{R_2 = R_2 / 3}
\begin{pmatrix}
2 & 1 & 1 \\
0 & 1 & 2 \\
0 & 0 & 0
\end{pmatrix}
\xrightarrow{R_1 = R_1 - R_2}
\begin{pmatrix}
1 & 1 & 1 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix}
\xrightarrow{R_3 = R_3 / 3}
\begin{pmatrix}
1 & 0 & -1 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]

We found \( c_1 = -1 \) and \( c_2 = 3 \). This means \( -b_1 + 3b_2 = u \), so \([u]_{\mathcal{B}} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}\).

c) From \([x]_{D} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}\), we have

\[
x = -d_1 + 3d_2 = -\begin{pmatrix} -2 \\ 1 \end{pmatrix} + 3 \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} + \begin{pmatrix} 9 \\ 3 \end{pmatrix} = \begin{pmatrix} 11 \\ 2 \end{pmatrix}.
\]